Higher order Polynomial Technique of Measuring Interest Rate Intensity: A Bridge Between Life Insurance Valuation and Investment

Colombo Economic Journal (CEJ)
Volume 3 Issue 2, December 2025: PP 69-92.
ISSN 2950-7480 (Print)
ISSN 2961-5437 (Online)
Copyright: © 2025 The Author(s)
Published by Department of Economics, University of Colombo, Sri Lanka
Website: https://arts.cmb.ac.lk/econ/colombo-economic-journal-cej/

G. M. Ogungbenle[1], W. Sirisena[2], C. Ukwu[2], J. S. Adeyele[1]

[1]Department of Actuarial Science, Faculty of Management Sciences, University of Jos, Nigeria

[2]Department of Mathematics, Faculty of Natural Sciences, University of Jos, Nigeria

Corresponding email: moyosiolorun@gmail.com

Received: 14 May 2025, **Revised**: 13 November, 2025, **Accepted**: 23 November 2025.

Abstract:

The current method of estimating the force of interest does not have the potential to account for small interest rates; hence it is less suitable for actuarial valuations. The paper attempts to suggest refinements to the existing method to capture interest rate dynamics accurately for actuarial valuation as the objective. The empirical underpinnings of instantaneous interest rate model are explored to conduct deep stress analysis through model equations and assess the degree to which the change in interest rate intensities can influence the long-run outcomes of some insurer's financial instruments. The findings provide insight into the robustness of these products in diverse interest rate environments, highlighting key factors that govern their sensitivity. Although, both the estimated and its limiting values coincide within an equilibrium interval $20 \le M \le 40$*, computational evidence from the results reveals that the trajectory of the limiting interest rate intensity within* $0 \le M < 5$ *is downward-sloping, representing a decreasing interest rate intensity over time while the trajectory of the estimated intensity is a curvature within* $M \ge 5$*. The force of interest's convergence to a stable 5% reflects the modelling equations' resilience and the dominance for a sustained growth factors across time. This type of behavior is characteristic of long-term life insurance instruments such as pensions or annuities where early volatility gives way to stable, compounding growth over decades. It is therefore recommended that the modelling equations be adopted by pricing actuaries when conducting interest rate sensitivities as the investment year progresses.*

Keywords: *Refinements, Valuation, Instantaneous, Robustness, Convergence, Intensities*

JEL Classification: C1, C4, C

Introduction

The current method of estimating the force of interest λ in actuarial finance and in financial engineering does not have the potential to account for small interest rates, nonlinear changes in rates, speculative financial instruments or sudden rate changes, hence it is less suitable for actuarial valuations. To the best of our knowledge, the current form $\log_e(1+i)$ where i is the interest rate and commonly applied in estimating the interest rate intensities in actuarial valuation has not been analytically justified and hence cannot be regarded as a mathematical model. The inability of the current method to account for small interest rates, that is as it does not account for the limit $\lim_{i\to 0}\log_e(1+i)$, nonlinear changes in rates, speculative financial instruments or sudden rate changes makes it less suitable for actuarial valuations. Due to these significant evident limitations, a more sophisticated methodology such as Euler-Maclaurin's series which includes higher order terms, provides a more accurate and flexible technique for modelling interest rate intensity particularly in cases requiring precise adjustments for changing interest rates or long-term financial projections.

The motivation for the works reviewed are two-fold. Firstly, the massive fluctuations which impact the insurance markets commencing with the subprime underwriting and pricing crisis in the developing economies holistically sharpened the knowledge of how crucial the role of interest rates regime is for a stable economy by reason of the interdependencies, hence robust estimation technique is required for the drastic life contingencies. Secondly, the ubiquitous non-analytic representation of interest rate intensities in actuarial finance has to be critically investigated. Thus, the pricing crisis describes in a regrettably painful manner that decisive investment variables strongly fluctuate over time horizon, ruling out, too simplicity methodologies, which fail to justify the empirical actuarial valuation. Since extant literatures have rarely developed actuarial models that are deeply rooted in analytic functional depth for the interest rate intensities, this paper significantly contributes by (i) developing a rigorously analytic actuarial model for the functional form for the interest rate intensities (ii) suggesting robust refinements to the current form and (iii) introducing a higher order Bernoulli polynomial to capture interest rate trajectories accurately for actuarial valuations. The goal is to introduce analytic methodologies to capture interest rate dynamics accurately in future actuarial valuations. Consequently, the study examines how variations in interest intensities under complex or constant instantaneous interest rate frameworks empirically compare with limiting its function.

The interest rate intensity plays a crucial role in valuing actuarial future liabilities, determining insurance premiums and evaluating the financial health of pension plans or other long-term financial products. Actuaries use the force of interest as a tool for continuous discounting which reflects the time value of money and ensures that insurance cash flows are appropriately adjusted for the effects of interest over time. Kasozi and Paulsen (2005) used the force of interest to model the insurance ruin probability function by solving the linear Volterra integral equation to generate value function but the functional form for the interest rate intensity was not developed. However, Udoye et al. (2021) applied step-wise extension of Vasicek model to jump diffusion model using Ito's formula in modelling interest rates in the presence of discontinuities. In constructing models for Tabarru funds using the Makeham's law, Muzaki, Siswanah and Miasary (2020) applied the force of interest to estimate the cost of insurance while Kutub et al. (2011) applied present value function with specified interest rates to estimate life annuity and term insurance. However, the functional form for the interest rate intensity was not analytically developed by the authors.

The interest rate intensity $\lambda(\xi)$ is the instantaneous rate of growth of an amount assuming continuous compounding. It is applied to model how amounts grow or shrink over time in financial models where interest is accrued continuously rather than at discrete intervals. In actuarial valuation, the force of interest assists to compute the present value of future liabilities, like insurance claims or pension benefits as well as determine how premiums could be set for insurance contracts. Castellares, Patricio and Lemonte (2022) applied a fixed force of interest to prepare mortality table under the framework of Makeham's mortality law. Kusumawati et al. (2024) applied lognormal distribution to model interest rates at random. Zhang (2007) estimated net single premium under random interest rates. The force of interest is central in the valuation of life insurance policies where future insurance benefits is discounted to the present (Luptakova & Bilikova, 2014). Mircea and Covrig (2015) and Aalaei (2022) both applied fuzzy interest rates to model annuity pricing under uncertainty. The mathematical derivations for the interest rate intensity under algorithmic expansion was not within their scope.

According to Liu (2010), Wu, Lin and Wang (2013) and Janardana and Wiriandi (2024), life insurers usually determine the present value of the future policy benefits that will be paid out upon the death of the insured. For insurance products, the premium is often estimated as the present value of the future insurance liabilities, which are determined based on expected benefits and expenses. The premiums are set such that they are sufficient to cover the insurer's obligations. To estimate the premium for a life insurance policy, actuaries use the following approach: (i) the expected benefit payments are modeled as random cash flows occurring at future times (ii) the expected future liabilities are discounted to the present using the force of interest. Once the present value of future liabilities is determined, actuaries compute the premium required to ensure that the insurers meet their obligations.

This premium must cover both the cost of the insurance benefits and the expenses of the insurer, adjusted for the time value of money using the force of interest. In pension planning, actuaries apply the force of interest to estimate the present value of pension obligations, that is the amount needed to be set aside today to cover future pension payments. In pension plan valuations, the force of interest is used to determine how the pension plan's liabilities grow and how the invested assets should grow to meet these obligations. The present value of future liabilities or obligations is a key component in actuarial valuations for products like annuities, life insurance and pension plans. These liabilities are discounted to the present using a force of interest to reflect the time value of money. Actuaries often deal with long-term cash flows such as those in pension plans, investment funds or annuities where payments are made at regular intervals. The force of interest is used to continuously discount these payments to the present. In reality, the force of interest is rarely constant. Actuaries account for this by using a varying force of interest in their models, based on changing interest rates or inflation expectations. This variability can significantly affect the present value calculations, impacting the determination of premiums, reserves and other actuarial calculations. As a result of this observation and to overcome this problem, it is assumed that the force of interest is constant. In the literature, there is none that develops model for the force of interest as applicable in life insurance.

Let p be the constant cashflow payments at integer time steps s where $s \in \{1, 2, 3, ..., m\}$. Then the present value of these payments is $PV_s = \sum_{s=1}^{m} pe^{-\delta s}$. The Euler-Maclaurin series for a smooth function $f(\xi)$ truncated to one correction term is defined as

$$\sum_{s=1}^{m} f(s) = \int_{1}^{m} f(\xi) d\xi + \frac{1}{2}\left(f(m) + f(1)\right) + \frac{1}{12}\left(f'(m) - f'(1)\right) \tag{1}$$

$$f(\xi) = pe^{-\delta\xi} \Rightarrow f'(\xi) = -p\delta e^{-\delta\xi} \tag{2}$$

$$\int_{1}^{m} f(\xi) d\xi = \int_{1}^{m} pe^{-\delta s} d\xi = \left[-\frac{p}{\delta} e^{-\delta s}\right]_{1}^{m} = \left(\frac{p}{\delta} e^{-\delta} - \frac{p}{\delta} e^{-\delta m}\right) \tag{3}$$

Substitute (2), and (3) into the series (1) yields

$$\sum_{s=1}^{m} f(s) = \left[\frac{p}{\delta}e^{-\delta} - \frac{p}{\delta}e^{-\delta m}\right] + \frac{1}{2}\left(pe^{-\delta m} + pe^{-\delta}\right) + \frac{1}{12}\left(-p\delta e^{-\delta m} + p\delta e^{-\delta}\right) \quad (4)$$

$$PV_{\delta} = \frac{p}{\delta}\left[e^{-\delta} - e^{-\delta m}\right] + \frac{p}{2}\left(e^{-\delta m} + e^{-\delta}\right) - \frac{p\delta}{12}\left(e^{-\delta m} - e^{-\delta}\right) \quad (5)$$

Define $U = e^{-\delta}$ and $V = e^{-\delta m}$, then the present value function is given as

$$\begin{aligned} PV_{s} &= \frac{p}{\delta}(U - V) + \frac{p}{2}(V + U) - \frac{p\delta}{12}(V - U) \\ &= p\left(\frac{U - V}{\delta} + \frac{V + U}{2} - \delta\left(\frac{V - U}{12}\right)\right) \end{aligned} \quad (6)$$

Unless the present value function PV_{δ} is known from the observed market price of an instrument such as level annuity, bond, pension obligation or insurance liability, equation (1h) is intractable to solve.

Material and Methods

Following Goldstein and Lee (2020) and Narlitasari et al. (2022), if d_y defines the number deaths between ages y and $y+1$, then the uniform distribution of death assumption will have ξd_y deaths per unit time for $0 < \xi < 1$ and as a result, the number of lives surviving at age $l_{y+\xi}$ is given by

$$l_{y+\xi} = l_y - \xi d_y \Rightarrow l_y - \xi\left(l_y - l_{y+1}\right) \quad (7)$$

$$l_{y+\xi} = l_y - \xi l_y + \xi l_{y+1} = (1-\xi)l_y + \xi l_{y+1} \quad (8)$$

The continuous whole life insurance function $\bar{A}_y$ is defined by

$$\bar{A}_y = \frac{-1}{l_y}\int_{0}^{\infty}\left(\frac{1}{1+i}\right)^{\xi} l'_{y+\xi}d\xi = \frac{-1}{l_y}\sum_{\theta=0}^{\infty}\int_{\theta}^{\theta+1}\left(\frac{1}{1+i}\right)^{\xi} l'_{y+\xi}d\xi \quad (9)$$

Let $\eta = \xi - \theta \Rightarrow \eta + \theta = \xi \Rightarrow d\eta = d\xi$ (10)

$$\bar{A}_y = \frac{-1}{l_y}\sum_{\theta=0}^{\infty}\int_{0}^{1}\left(\frac{1}{1+i}\right)^{\eta+\theta} l'_{y+\eta+\theta}d\eta \quad (11)$$

Using the uniform distribution of death assumption on l_y, we obtain

$$l_{y+\theta+\eta} = (1-\eta)l_{y+\theta} + \eta l_{y+\theta+1} \Rightarrow \frac{d}{d\eta}l_{y+\theta+\eta} = -l_{y+\theta} + l_{y+\theta+1} \quad (12)$$

$$\bar{A}_y = \frac{-1}{l_y}\sum_{\theta=0}^{\infty}\int_0^1\left(\frac{1}{1+i}\right)^{\eta+\theta}\frac{d}{d\eta}l_{y+\theta+\eta}d\eta = \frac{-1}{l_y}\sum_{\theta=0}^{\infty}\int_0^1\left(\frac{1}{1+i}\right)^{\eta+\theta}\left(-l_{y+\theta}+l_{y+\theta+1}\right)d\eta \quad (13)$$

$$\bar{A}_y = \sum_{\theta=0}^{\infty}\int_0^1\left(\frac{1}{1+i}\right)^{\eta+\theta}\frac{\left(l_{y+\theta}-l_{y+\theta+1}\right)}{l_y}d\eta = \sum_{\theta=0}^{\infty}\left(\frac{1}{1+i}\right)^{\theta}\frac{\left(l_{y+\theta}-l_{y+\theta+1}\right)}{l_y}\int_0^1\left(\frac{1}{1+i}\right)^{\eta}d\eta$$

(14)

$$\bar{A}_y = \sum_{\theta=0}^{\infty}v^{\theta}\left(\frac{l_{y+\theta}}{l_y}-\frac{l_{y+\theta+1}}{l_y}\right)\int_0^1 v^{\eta}d\eta = \sum_{\theta=0}^{\infty}v^{\theta}\left({}_{\theta}p_y - {}_{\theta+1}p_y\right)\int_0^1 v^{\eta}d\eta \quad (15)$$

$$\bar{A}_y = \sum_{\theta=0}^{\infty}v^{\theta}\left({}_{\theta|1}q_y\right)\left(\int_0^1 v^{\eta}d\eta\right) = \sum_{\theta=0}^{\infty}v^{\theta}\left({}_{\theta|1}q_y\right)\left(\int_0^1 e^{-\lambda\eta}d\eta\right) \quad (16)$$

$$\bar{A}_y = \sum_{\theta=0}^{\infty}v^{\theta}\left({}_{\theta|1}q_y\right)\left[\frac{e^{-\lambda\eta}}{-\lambda}\right]_0^1 = \sum_{\theta=0}^{\infty}v^{\theta}\left({}_{\theta|1}q_y\right)\left(\frac{e^{-\lambda}}{-\lambda}-\frac{1}{-\lambda}\right) \quad (17)$$

$$\bar{A}_y = \sum_{\theta=0}^{\infty}v^{\theta}\left({}_{\theta|1}q_y\right)\left(\frac{e^{-\lambda}}{-\lambda}-\frac{1}{-\lambda}\right) = \sum_{\theta=0}^{\infty}v^{\theta}\left({}_{\theta|1}q_y\right)\left(\frac{1-e^{-\lambda}}{\lambda}\right) \quad (18)$$

$$\bar{A}_y = \sum_{\theta=0}^{\infty}v^{\theta}\left({}_{\theta|1}q_y\right)\left(\frac{1-v}{\lambda}\right) = \sum_{\theta=0}^{\infty}v^{\theta}\left({}_{\theta|1}q_y\right)\left(\frac{d}{\lambda}\right) \quad (19)$$

$$\bar{A}_y = \sum_{\theta=0}^{\infty}v^{\theta}\left({}_{\theta|1}q_y\right)\left(\frac{1-v}{\lambda}\right) = \sum_{\theta=0}^{\infty}v^{\theta}\left({}_{\theta|1}q_y\right)\left(\frac{iv}{\lambda}\right) \quad (20)$$

$$\bar{A}_y = \sum_{\theta=0}^{\omega-y-1}v^{\theta}\left({}_{\theta|1}q_y\right)\left(\frac{1-v}{\lambda}\right) = \left(\frac{i}{\lambda}\right)\sum_{\theta=0}^{\omega-y-1}v^{\theta+1}\left({}_{\theta|1}q_y\right) = \left(\frac{i}{\lambda}\right)A_y \quad (21)$$

Recall that in A_x, payment is delayed till the end of the year and to account for the effect of this delay, the present value function is multiplied by $e^{-\lambda\xi}$ where ξ represents the period through which payment is delayed.

$$i = e^{\lambda}-1 \quad (22)$$

$$\bar{A}_y = \left(\frac{e^{\lambda}-1}{\lambda}\right)A_y \Rightarrow \frac{A_y}{\bar{A}_y} = \frac{\lambda}{e^{\lambda}-1} \quad (23)$$

recall that,

$$\lambda\times\bar{a}_{\overline{M}|} = i\times a_{\overline{M}|} \quad (24)$$

Hence

$$\frac{M}{a_{\overline{M}|}} = \frac{M\times i}{\left(a_{\overline{M}|}\right)\times i} = \frac{M\left(e^{\lambda}-1\right)}{\left(\bar{a}_{\overline{M}|}\right)\times\lambda} = \frac{M}{\left(\bar{a}_{\overline{M}|}\right)}\times\frac{\left(e^{\lambda}-1\right)}{\lambda} \quad (25)$$

$$\frac{M}{a_{\overline{M|}}}=\frac{M}{\bar{a}_{\overline{M|}}}\times\left\{\frac{e^{\lambda}-1}{\lambda}\right\}=\frac{M}{\bar{a}_{\overline{M|}}}\times\frac{\bar{A}_y}{A_y} \tag{26}$$

$$\frac{M}{a_{\overline{M|}}}=\frac{M}{\bar{a}_{\overline{M|}}}\times\left\{\frac{e^{\lambda}-1}{\lambda}\right\}=\frac{M}{\left\{\frac{1-e^{-M\lambda}}{\lambda}\right\}}\times\left\{\frac{e^{\lambda}-1}{\lambda}\right\}=\frac{M\lambda}{\left(1-e^{-M\lambda}\right)}\times\left\{\frac{e^{\lambda}-1}{\lambda}\right\} \tag{27}$$

$$=\frac{\bar{A}_y}{A_y}\times\frac{-M\lambda}{\left(e^{-M\lambda}-1\right)}$$

The last term in (27) can be written as

$$\frac{M\lambda}{\left(1-e^{-M\lambda}\right)}\times\left\{\frac{e^{\lambda}-1}{\lambda}\right\}=\frac{-M\lambda}{\left(e^{-M\lambda}-\right)}\times\left\{\frac{e^{\lambda}-1}{\lambda}\right\}=\frac{1}{\left\{\frac{\lambda}{e^{\lambda}-1}\right\}}\times\left\{\frac{-M\lambda}{e^{-M\lambda}-1}\right\} \tag{28}$$

The Euler-Maclaurin Series

$$\frac{f(0)+f(1)}{2}=\int_0^1 f(y)\,dy+\left[f^{(1)}(1)-f^{(1)}(0)\right]\frac{B_2}{2!}$$

$$+\left[f^{(3)}(1)-f^{(3)}(0)\right]\frac{B_4}{4!}+\ldots+\left[f^{(2M-1)}(1)-f^{(2M-1)}(0)\right]\frac{B_{2M}}{(2M)!} \tag{29}$$

$$-\int_0^1 f^{(2M)}(y)\frac{B_{2M}(y)}{(2M)!}dy$$

$M\geq 1$ is an integer and $f(y)$ has $2M$ continuous derivatives in $0\leq y\leq 1$. The B_r, $r=0,1,2,3,\ldots,2M$ are defined by

$$\frac{y}{e^{y}-1}=\sum_{r=0}^{\infty}\frac{B_r y^r}{r!} \tag{30}$$

$$B_0=1; B_1=-\frac{1}{2}; B_2=\frac{1}{6}; B_3=B_5=B_7=B_9=0; B_4=-\frac{1}{30};$$
$$B_6=\frac{1}{42}; B_8=-\frac{1}{30}; B_{10}=\frac{5}{66};\ldots \tag{31}$$

Suppose that $f(y)=h(y+1)$, then

$$h(1)+h(2)+h(3)+\ldots+h(M)=\int_1^M f(y)dy+\frac{1}{2}h(M)$$
$$+\frac{B_2}{2!}h'(M)+\frac{B_4}{4!}h'''(M)+\ldots+$$
$$+\frac{B_{2M}}{(2M)!}\left[h^{(2M-1)}(M)\right]-\int_1^M h^{(2M)}(y)\frac{B_{2M}(y-[y])}{(2M)!}dy \quad (32)$$
$$+\frac{1}{2}h(1)-\frac{B_2}{2!}h'(1)-\ldots\frac{B_{2M}}{(2M)!}\left[h^{(2M-1)}(1)\right]$$

The function $h(y)$ has $2M$ continuous derivatives whenever $x \geq 1$

$$B_{2M}(y-[y])=2(2M)!(2\pi)^{-2M}(-1)^{M+1}\sum_{j=1}^{\infty} j^{-2M}\cos(2j\pi y) \quad (33)$$

If $M=1,2,3,\ldots$, it follows that

$$\left|B_{2M}(x-[x])\right| \leq \left|B_{2M}\right| = 2(2k)!(2\pi)^{-2M}\sum_{j=1}^{\infty} j^{-2M} \quad (34)$$

The last term in equation (28) mirrors the Bernoulli polynomial (30) and the expansion is given in equation (35)

$$\frac{1}{\left\{\frac{\lambda}{e^{\lambda}-1}\right\}}\times\left\{\frac{-M\lambda}{e^{-M\lambda}-1}\right\}\cong$$
$$\frac{\left\{\begin{array}{l}1+\frac{1}{2}M\lambda+\frac{1}{12}M^2\lambda^2-\frac{1}{720}M^4\lambda^4+\frac{1}{30240}M^6\lambda^6 \\ -\frac{1}{1209600}M^8\lambda^8 \\ +\frac{1}{47900160}M^{10}\lambda^{10}+\ldots\end{array}\right\}}{\left\{\begin{array}{l}1-\frac{1}{2}\lambda+\frac{1}{12}\lambda^2-\frac{1}{720}\lambda^4+\frac{1}{30240}\lambda^6-\frac{1}{1209600}\lambda^8 \\ +\frac{1}{47900160}\lambda^{10}+\ldots\end{array}\right\}} \quad (35)$$

However, when $M=-1$ in the numerator of (35) yields

$$\frac{\lambda}{e^{\lambda}-1} \cong 1-\frac{1}{2}\lambda+\frac{1}{12}\lambda^{2}-\frac{1}{720}\lambda^{4}+\frac{1}{30240}\lambda^{6}-\frac{1}{1209600}\lambda^{8} + \frac{1}{47900160}\lambda^{10}+\ldots \tag{36}$$

Consider the quartic estimation and ignore the fifth term and above in the numerator and denominator of (35).

$$\frac{M}{a_{\overline{M}|}} \cong \frac{1+\frac{1}{2}M\lambda+\frac{1}{12}M^{2}\lambda^{2}-\frac{1}{720}M^{4}\lambda^{4}}{1-\frac{1}{2}\lambda+\frac{1}{12}\lambda^{2}-\frac{1}{720}\lambda^{4}} \tag{37}$$

Subtracting 1 from both sides of (37),

$$\frac{M}{a_{\overline{M}|}}-1 \cong \frac{1+\frac{1}{2}M\lambda+\frac{1}{12}M^{2}\lambda^{2}-\frac{1}{720}M^{4}\lambda^{4}}{1-\frac{1}{2}\lambda+\frac{1}{12}\lambda^{2}-\frac{1}{720}\lambda^{4}}-1 \tag{38}$$

$$\frac{M}{a_{\overline{M}|}}-1 \cong \frac{1+\frac{1}{2}M\lambda+\frac{1}{12}M^{2}\lambda^{2}-\frac{1}{720}M^{4}\lambda^{4}-\left(1-\frac{1}{2}\lambda+\frac{1}{12}\lambda^{2}-\frac{1}{720}\lambda^{4}\right)}{1-\frac{1}{2}\lambda+\frac{1}{12}\lambda^{2}-\frac{1}{720}\lambda^{4}} \tag{39}$$

$$\frac{M}{a_{\overline{M}|}}-1 \cong \frac{1+\frac{1}{2}M\lambda+\frac{1}{12}M^{2}\lambda^{2}-\frac{1}{720}M^{4}\lambda^{4}-1+\frac{1}{2}\lambda-\frac{1}{12}\lambda^{2}+\frac{1}{720}\lambda^{4}}{1-\frac{1}{2}\lambda+\frac{1}{12}\lambda^{2}-\frac{1}{720}\lambda^{4}} \tag{40}$$

$$\theta=\frac{M}{a_{\overline{M}|}}-1=\frac{M-a_{\overline{M}|}}{a_{\overline{M}|}} \tag{41}$$

$$\Rightarrow \theta=\frac{\frac{1}{2}M\lambda+\frac{1}{12}M^{2}\lambda^{2}-\frac{1}{720}M^{4}\lambda^{4}+\frac{1}{2}\lambda-\frac{1}{12}\lambda^{2}+\frac{1}{720}\lambda^{4}}{1-\frac{1}{2}\lambda+\frac{1}{12}\lambda^{2}-\frac{1}{720}\lambda^{4}} \tag{42}$$

$$\Rightarrow \theta\left(1-\frac{1}{2}\lambda+\frac{1}{12}\lambda^{2}-\frac{1}{720}\lambda^{4}\right) = \frac{1}{2}M\lambda+\frac{1}{12}M^{2}\lambda^{2}-\frac{1}{720}M^{4}\lambda^{4}+\frac{1}{2}\lambda-\frac{1}{12}\lambda^{2}+\frac{1}{720}\lambda^{4} \tag{43}$$

$$\Rightarrow \theta - \frac{1}{2}\theta\lambda + \frac{1}{12}\theta\lambda^2 - \frac{1}{720}\theta\lambda^4$$
$$= \frac{1}{2}M\lambda + \frac{1}{12}M^2\lambda^2 - \frac{1}{720}M^4\lambda^4 + \frac{1}{2}\lambda - \frac{1}{12}\lambda^2 + \frac{1}{720}\lambda^4 \quad (44)$$

$$\Rightarrow \frac{1}{720}M^4\lambda^4 - \frac{1}{720}\theta\lambda^4 - \frac{1}{720}\lambda^4 - \frac{1}{12}M^2\lambda^2$$
$$+\frac{1}{12}\theta\lambda^2 + \frac{1}{12}\lambda^2 - \frac{1}{2}M\lambda - \frac{1}{2}\theta\lambda - \frac{1}{2}\lambda + \theta = 0 \quad (45)$$

$$\Rightarrow M^4\lambda^4 - \theta\lambda^4 - \lambda^4 - 60M^2\lambda^2 + 60\theta\lambda^2$$
$$+60\lambda^2 - 360k\lambda - 360\theta\lambda - 360\lambda + 720\theta = 0 \quad (46)$$

$$\Rightarrow \left(M^4 - \theta - 1\right)\lambda^4 - \left(60M^2 - 60\theta - 60\right)\lambda^2$$
$$-\left(360M + 360\theta + 360\right)\lambda + 720\theta = 0 \quad (47)$$

Substitute (41) into (47) to get

$$\Rightarrow \left(M^4 - \left(\frac{M - a_{\overline{M|}}}{a_{\overline{M|}}}\right) - 1\right)\lambda^4 - \left(60M^2 - 60\left(\frac{M - a_{\overline{M|}}}{a_{\overline{M|}}}\right) - 60\right)\lambda^2$$
$$-\left(360M + 360\left(\frac{M - a_{\overline{M|}}}{a_{\overline{M|}}}\right) + 360\right)\lambda + 720\left(\frac{M - a_{\overline{M|}}}{a_{\overline{M|}}}\right) = 0 \quad (48)$$

This equation (47) is of the form;

$$a_1\lambda^4 + a_2\lambda^3 + a_3\lambda^2 + a_4\lambda + a_5 = 0 \quad (49)$$

$a_1 = \left(M^4 - \theta - 1\right)$; $a_2 = 0$; $a_3 = -\left(60M^2 - 60\theta - 60\right)$;

$a_4 = -\left(360M + 360\theta + 360\right)$; $a_5 = 720\theta$. To study the behavior of the equation in the long-run, we extend period M as far as 100 to represent infinity. The modelling equations were derived in Appendix A and are based on 5% rate of interest. The solution to the modelling equations in in table 1 are presented in Appendix B.

Discussion of Results

The modelling equations (Appendix A) present a rigorous empirical description where the growth of an insurance fund over time is described by a polynomial of high degree capturing complex investment dynamics. This is because insurer’s financial systems can be characterised by a non-linear and chaotic manner, such that small changes in the underlying variables or initial conditions could lead to oscillatory behavior. This occurs where interest rates are influenced by a combination of factors which interact in a non-linear manner. When this evolves, the force of interest will not assume a simple monotone pattern but instead may exhibit oscillations as a reflection of the underlying complexity. The modelling equations obtained represent an investment environment where different forces both speculative (complex) and real interact to produce equilibrium. Obviously In table 1 (Appendix B), it is interesting to observe that the real rates in root 1, and the real parts of the complex rates in both root2, root3, add up to root4, in absolute values that is, $root1 + real\ part\ of\ \mathrm{root}2 + real\ part\ of\ \mathrm{root}3 = root4$ while the sum of the imaginary rates frizzle out. The real part represents actual economic factors while the complex parts represent the speculative influences. The real roots are the observed values such as stable interest rates or growth factors in the economic modelling equations. However, the complex roots which conjugate each other define some more theoretical factors such as periodic investment conditions that may not have a straight forward market equivalence but still impact the financial system.

The zeroes obtained evolve from solving the underlying fourth-degree equation governing the interest rate intensities over a 100-year investment horizon. The rows correspond to an investment period $M = k$, with four zeroes of the modelling equations but the resulting limiting value of interest rate intensities is in the last column of the table. The behavior of these roots over time reflects the evolving dynamics of the insurer's financial system being modelled. Root 1 displays an interesting metamorphosis across the investment horizon. At $M = 1$, the intensity is a negative real number, representing an initially loss-driven or unfavorable driven investment condition. From $M = 2$ to $M = 51$, the interest rate (root 1) becomes positive real number, and suggests a shift into a favorable investment condition in a period of real and stable growth. However, from $M = 52$,upwards, root 1 becomes negative complex numbers, indicating the commencement of oscillatory behavior with a damping effect. Consequently, this transition may result in long-run instability or changing market dynamics, possibly associated with policy changes or macroeconomic factors governing the investment at the long run.

Root 2 commences as a positive complex number at $M = 1$, which suggests initial volatility with growth potential-characteristic of speculative or emerging investment conditions. However, from $M = 2$, to $M = 100$, root 2 becomes negative complex number. This adverse shift indicates there is an auxiliary component of the financial system describing an economic drag and causing continuous cyclical decline. The economic drag may have evolved due to the systemic risks, market inefficiencies. Consequently, these impact negatively to the interest rate intensity throughout the investment period.

Root 3 resembles varying market movement. It starts as a positive complex number at $M = 1$, similar to root 2, indicating early volatility. However, it progressively shifts to a negative complex number between $M = 2$ and $M = 51$ representing an extended period of decline governed by cyclical behavior. Interestingly, from $M = 52$ to $M = 100$, root 3 becomes a positive real number. This suggests a component within the financial system, after a period of instability, which transforms into a stable level of positive growth, possibly a maturing investment that seems to admit long-term value structural adjustment. Root 4 remains real and positive throughout the entire 100-year period. Its consistent positive values admit that it is the dominant intensity in the system, suggesting a stable and reliable driver of the interest rates. This root may likely reinforce the crucial strength of the investment modelling equations, probably a basic economic factor or policy mechanism, which ensures continuity and long-term stability.

The final column, defining the limiting value of the force of interest, represent a slow but steady convergence towards approximately 5% annually. Although it initially fell slightly below this mark, the values grow upward as M increases, indicating a level of diminishing effect of the volatile or complex roots. Across time, the interest rate intensities stabilise in line with the behavior of root 4. This convergence indicates that in spite of short-term volatility and mid-term complexity, the investment environment being modelled, attains a stable and predictable level at the long run. The modelling equations is a replica of an investment environment associated with instability or possible losses, transits through a period of recovery and growth, encounters speculative cyclical influences and finally settles into long-term equilibrium. The interest rate convergence to a stable 5% is a proof of the insurer's system's resilience and the dominance of sustained growth factors over time. This manner of behavior characterises long-term financial instruments such as pensions or annuities where early volatility gives way to stable, compounding growth over long period of time. In view of Stehlik et al. (2024), during times of negative interest rates or unusual market conditions, the force of interest may be represented as a negative and or a complex number.

Figure 1 column 4 and 5 (Appendix C) compares the estimated interest rate intensity with the limiting value. The limiting values of the interest rate intensity may not be obtainable in investment because its trajectory defines a straight line. In Figure 1, both the trajectory of the limiting value and the trajectory of the estimated value intersect within $20 \le M \le 40$ called the equilibrium interval. In figure 1, the trajectory of the estimated force of interest is a curve indicating that the rate of interest is changing continuously over time. The curvature provides insights into how the rate of interest evolves whether it is accelerating or decelerating. Obviously, the interest rate intensity is increasing over time. This means that the rate of interest is getting higher as time increases. The implication is that as the force of interest increases, the growth of the investment or accumulation of value will accelerate because the rate at which interest is being applied is rising. This defines a condition where the market interest rate is rises steadily over time or a condition where demand for a particular asset increases over time causing interest rates to rise. The smooth curvature demonstrates that the model equations are robust to capture variations in the market.

Consequently, the curve of the limiting interest rate intensity within $0 \le M < 5$ is downward-sloping representing a decreasing interest over time. This suggests that the interest rate is becoming lower as time progresses. As the force of interest decreases, the growth of the investment will slow down since the rate at which interest is added to the investment is reducing. This might occur in a situation where inflation is expected to decrease or where Central Bank reduces interest rates over time. As $M > 5$, the curve becomes a straight line. The force of interest is changing at a constant rate indicating that it is either linearly increasing or decreasing over time. The interest rate is either rising or falling at a consistent rate causing the investment's growth to follow a predictable pattern.

Hoang (2015) constructed the following ruin probabilities under constant interest rate intensity

$$U_\lambda(\xi) = ue^{\lambda\xi} + \pi\int_0^\xi e^{\lambda s}ds - S(\xi)e^{\lambda\xi} = ue^{\lambda\xi} + \pi\left(\frac{e^{\lambda\xi}-1}{\lambda}\right) - S(\xi)e^{\lambda\xi} \quad (50)$$

where $U_\lambda(\xi)$ is the surplus up till time ξ, π the premium rate, $S(\xi) = \sum_{m=1}^{M(\xi)} Y_m$ is the aggregate claim, $M(\xi) = Sup\{m | T_m \le \xi\}$ is the number of claims, $T_m = \sum_{k=1}^{m} \xi_k$ is the time of the mth claim and $U_\lambda(0) = u$ but there was not any analytic evidence that the author developed any analytic model for the force of interest and hence assumed $\lambda = 0.1$

Under some investment conditions, interest rates or the forces driving interest rate changes may be subject to periodic fluctuations as a result of the recurring investment conditions, such as the speculative rise and fall of economic activity in a market economy, including expansion and contraction phases. These cycles can result in oscillating interest rates as the Central Bank modifies its policies in response to the subsisting economic environments. Inflation rates may rise or fall periodically, impacting the interest rates. However, deflationary pressures may lead to oscillations in the effective rate of interest. Central banks often adjust interest rates in response to economic conditions, intending to contain inflation, manage unemployment or stimulate growth. These rate adjustments may not be steady as they could occur in a more stepwise or oscillatory manner in response to changing economic conditions. If a Central Bank's policy involves periodic interventions, such as adjusting interest rates every periodically in connection to a particular economic problem, the interest rate intensities may be characterised by oscillations.

Investors may react to the perceived changes in risk, socio-political contingencies or other market shocks. As a result, the investment markets are not only driven by the forces of supply and demand but also by the speculative investor's sentiment and behavioral responses. This may result in oscillations in the effective interest rates as the market adjusts expectations about future growth, inflation or risk.
External shocks, such as socio-political events, or sudden economic crises, may cause volatility in interest rates in the interim. If these shocks occur periodically or in a manner which impacts the economy cyclically, they may cause the force of interest to oscillate. When there are periods of high investment or borrowing demand, interest rates may rise and if demand declines, interest rates might fall. Consequently, if this demand fluctuates with some level of regularity such as seasonal demand for capital or cyclical shifts in investment patterns, the force of interest could exhibit oscillations. Investors' expectations about future interest rates or economic conditions can cause oscillations in the interest rate intensities. If investors anticipate future bearish or bullish investment conditions, their expectations can lead to oscillatory changes in short-term interest rates, which in turn influence the force of interest. Behavioral factors such as herd's behavior can introduce oscillations in financial markets. Herd-behaviour suggests that investors tend to imitate the financial behaviour of the majority of the herd. Herding is widely acknowledged in the investment market as the reason behind dramatic rallies and sell-offs.

Conclusion

In conclusion, this study has explored the sensitivity of instantaneous interest rate in comparison with its limiting functions. By analysing the sensitivity of these financial instruments to variations in instantaneous interest rates, the study provides valuable insights into the potential risks and opportunities for life insurers, investors and actuarial scientists. The results suggest that a deeper understanding of instantaneous interest rate movements is essential for the valuation of annuity pricing and accumulation strategies especially in volatile or uncertain financial climates. Moreover, the study emphasises the need for improved models that account for such sensitivities in order to enhance the robustness of financial planning and investment decision-making. Future research may focus on refining these models to incorporate more dynamic components, such as time-varying interest rate volatilities and alternative financial assumptions, to capture the complexities of the financial markets so that insurance practitioners can align their strategies within the evolving economic frameworks, ensuring more accurate pricing, risk management and investment outcomes.

Conflict of Interest

The authors declare that there is no conflict of interest

Acknowledgements: We wish to thank the anonymous reviewers and the Editorial Board for the constructive criticism on the earlier drafts.

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APPENDIX A

$$5\lambda^4 - 300\lambda^2 + 73800\lambda - 3600 = 0; \quad M = 1 \tag{A1}$$

$$1492\lambda^4 - 17546\lambda^2 - 110722\lambda + 5444 = 0; \quad M = 2 \tag{A2}$$

$$7990\lambda^4 - 47390\lambda^2 - 147659\lambda + 7317 = 0; \quad M = 3 \tag{A3}$$

$$25487\lambda^4 - 89232\lambda^2 - 184610\lambda + 9219 = 0; \quad M = 4 \tag{A4}$$

$$62385\lambda^4 - 143071\lambda^2 - 221575\lambda + 11151 = 0; \quad M = 5 \tag{A5}$$

$$129482\lambda^4 - 208907\lambda^2 - 258556\lambda + 13112 = 0; \quad M = 6 \tag{A6}$$

$$239979\lambda^4 - 286742\lambda^2 - 295551\lambda + 15101 = 0; \quad M = 7 \tag{A7}$$

$$409476\lambda^4 - 376573\lambda^2 - 332560\lambda + 17120 = 0; \quad M = 8 \tag{A8}$$

$$655973\lambda^4 - 478403\lambda^2 - 369584\lambda + 19167 = 0; \quad M = 9 \tag{A9}$$

$$999870\lambda^4 - 592230\lambda^2 - 406622\lambda + 21243 = 0; \quad M = 10 \tag{A10}$$

$$1463968\lambda^4 - 718054\lambda^2 - 443674\lambda + 23348 = 0; \quad M = 11 \tag{A11}$$

$$2073465\lambda^4 - 855877\lambda^2 - 480741\lambda + 25481 = 0; \quad M = 12 \tag{A12}$$

$$2855962\lambda^4 - 1005696\lambda^2 - 517821\lambda + 27643 = 0; \quad M = 13 \tag{A13}$$

$$3841459\lambda^4 - 1167514\lambda^2 - 554916\lambda + 29832 = 0; \quad M = 14 \tag{A14}$$

$$5062355\lambda^4 - 1341329\lambda^2 - 592025\lambda + 32050 = 0; \quad M = 15 \tag{A15}$$

$$6553452\lambda^4 - 1527142\lambda^2 - 629147\lambda + 34295 = 0; \quad M = 16 \tag{A16}$$

$$8351949\lambda^4 - 1724953\lambda^2 - 666284\lambda + 36568 = 0; \quad M = 17 \tag{A17}$$

$$10497446\lambda^4 - 1934761\lambda^2 - 703434\lambda + 38868 = 0; \quad M = 18 \tag{A18}$$

$$13031943\lambda^4 - 2156567\lambda^2 - 740598\lambda + 41195 = 0; \quad M = 19 \tag{A19}$$

$$15999840\lambda^4 - 2390371\lambda^2 - 777775\lambda + 43549 = 0; \quad M = 20 \tag{A20}$$

$$19447936\lambda^4 - 2636172\lambda^2 - 814965\lambda + 45930 = 0; \quad M = 21 \qquad \text{(A21)}$$

$$23425433\lambda^4 - 2893972\lambda^2 - 852169\lambda + 48337 = 0; \quad M = 22 \qquad \text{(A22)}$$

$$27983929\lambda^4 - 3163769\lambda^2 - 889385\lambda + 50771 = 0; \quad M = 23 \qquad \text{(A23)}$$

$$33177426\lambda^4 - 3445564\lambda^2 - 926615\lambda + 53230 = 0; \quad M = 24 \qquad \text{(A24)}$$

$$39062323\lambda^4 - 3739357\lambda^2 - 963857\lambda + 55714 = 0; \quad M = 25 \qquad \text{(A25)}$$

$$45697419\lambda^4 - 4045148\lambda^2 - 1001112\lambda + 58224 = 0; \quad M = 26 \qquad \text{(A26)}$$

$$53143916\lambda^4 - 4362937\lambda^2 - 1038380\lambda + 60759 = 0; \quad M = 27 \qquad \text{(A27)}$$

$$61465412\lambda^4 - 4692723\lambda^2 - 1075660\lambda + 63319 = 0; \quad M = 28 \qquad \text{(A28)}$$

$$70727908\lambda^4 - 5034508\lambda^2 - 1112952\lambda + 65903 = 0; \quad M = 29 \qquad \text{(A29)}$$

$$80999805\lambda^4 - 5388291\lambda^2 - 1150256\lambda + 68511 = 0; \quad M = 30 \qquad \text{(A30)}$$

$$92351901\lambda^4 - 5754071\lambda^2 - 1187571\lambda + 71143 = 0; \quad M = 31 \qquad \text{(A31)}$$

$$104857398\lambda^4 - 6131850\lambda^2 - 1224899\lambda + 73798 = 0; \quad M = 32 \qquad \text{(A32)}$$

$$118591894\lambda^4 - 6521627\lambda^2 - 1262238\lambda + 76476 = 0; \quad M = 33 \qquad \text{(A33)}$$

$$133633390\lambda^4 - 6923402\lambda^2 - 1299589\lambda + 79177 = 0; \quad M = 34 \qquad \text{(A34)}$$

$$150062286\lambda^4 - 7337175\lambda^2 - 1336950\lambda + 81901 = 0; \quad M = 35 \qquad \text{(A35)}$$

$$167961382\lambda^4 - 7762946\lambda^2 - 1374323\lambda + 84646 = 0; \quad M = 36 \qquad \text{(A36)}$$

$$187415879\lambda^4 - 8200716\lambda^2 - 1411707\lambda + 87413 = 0; \quad M = 37 \qquad \text{(A37)}$$

$$208513375\lambda^4 - 8650483\lambda^2 - 1449101\lambda + 90202 = 0; \quad M = 38 \qquad \text{(A38)}$$

$$231343871\lambda^4 - 9112249\lambda^2 - 1486506\lambda + 93011 = 0; \quad M = 39 \qquad \text{(A39)}$$

$$255999767\lambda^4 - 9586013\lambda^2 - 1523921\lambda + 95841 = 0; \quad M = 40 \qquad \text{(A40)}$$

$$282575863\lambda^4 - 10071776\lambda^2 - 1561346\lambda + 98691 = 0; \quad M = 41 \qquad \text{(A41)}$$

$$311169359\lambda^4 - 10569537\lambda^2 - 1598781\lambda + 101562 = 0; \quad M = 42 \qquad \text{(A42)}$$

$$341879855\lambda^4 - 11079296\lambda^2 - 1636226\lambda + 104451 = 0; \quad M = 43 \qquad \text{(A43)}$$

$$374809351\lambda^4 - 11601053\lambda^2 - 1673680\lambda + 107360 = 0; \quad M = 44 \qquad \text{(A44)}$$

$$410062247\lambda^4 - 12134809\lambda^2 - 1711144\lambda + 110288 = 0; \quad M = 45 \qquad \text{(A45)}$$

$$447745343\lambda^4 - 12680564\lambda^2 - 1748617\lambda + 113234 = 0; \quad M = 46 \qquad \text{(A46)}$$

$$487967839\lambda^4 - 13238317\lambda^2 - 1786099\lambda + 116198 = 0; \quad M = 47 \qquad \text{(A47)}$$

$$530841334\lambda^4 - 13808068\lambda^2 - 1823590\lambda + 119180 = 0; \quad M = 48 \qquad \text{(A48)}$$

$$576479830\lambda^4 - 14389818\lambda^2 - 1861090\lambda + 122180 = 0; \quad M = 49 \qquad \text{(A49)}$$

$$624999726\lambda^4 - 14983567\lambda^2 - 1898598\lambda + 125196 = 0; \quad M = 50 \qquad \text{(A50)}$$

$$676519822\lambda^4 - 15589344\lambda^2 - 1936115\lambda + 128229 = 0; \quad M = 51 \qquad \text{(A51)}$$

$$731161318\lambda^4 - 16207060\lambda^2 - 1973639\lambda + 131279 = 0; \quad M = 52 \qquad \text{(A52)}$$

$$789047813\lambda^4 - 16836805\lambda^2 - 2011172\lambda + 134344 = 0; \quad M = 53 \qquad \text{(A53)}$$

$$850305309\lambda^4 - 17478548\lambda^2 - 2048712\lambda + 137425 = 0; \quad M = 54 \qquad \text{(A54)}$$

$$915062205\lambda^4 - 18132290\lambda^2 - 2086260\lambda + 140521 = 0; \quad M = 55 \qquad \text{(A55)}$$

$$9834449301\lambda^4 - 18798031\lambda^2 - 2123816\lambda + 143632 = 0; \quad M = 56 \qquad \text{(A56)}$$

$$1055599796\lambda^4 - 19475770\lambda^2 - 2161379\lambda + 146757 = 0; \quad M = 57 \qquad \text{(A57)}$$

$$1131649292\lambda^4 - 20165509\lambda^2 - 2198949\lambda + 149897 = 0; \quad M = 58 \qquad \text{(A58)}$$

$$1211735787\lambda^4 - 20867246\lambda^2 - 2236525\lambda + 153051 = 0; \quad M = 59 \qquad \text{(A59)}$$

$$1295999683\lambda^4 - 21580982\lambda^2 - 2274109\lambda + 156218 = 0; \quad M = 60 \qquad \text{(A60)}$$

$$1384583779\lambda^4 - 22306717\lambda^2 - 2311699\lambda + 159398 = 0; \quad M = 61 \qquad \text{(A61)}$$

$$1477633274\lambda^4 - 23044451\lambda^2 - 2349296\lambda + 162591 = 0; \quad M = 62 \quad \text{(A62)}$$

$$1575295770\lambda^4 - 23794184\lambda^2 - 2386899\lambda + 165797 = 0; \quad M = 63 \quad \text{(A63)}$$

$$1677721265\lambda^4 - 24555915\lambda^2 - 2424508\lambda + 169015 = 0; \quad M = 64 \quad \text{(A64)}$$

$$1785062161\lambda^4 - 25329646\lambda^2 - 2462123\lambda + 172245 = 0; \quad M = 65 \quad \text{(A65)}$$

$$1897473256\lambda^4 - 26115376\lambda^2 - 2499743\lambda + 175487 = 0; \quad M = 66 \quad \text{(A66)}$$

$$2015111752\lambda^4 - 26913105\lambda^2 - 2537370\lambda + 178740 = 0; \quad M = 67 \quad \text{(A67)}$$

$$2138137247\lambda^4 - 27722833\lambda^2 - 2575002\lambda + 182004 = 0; \quad M = 68 \quad \text{(A68)}$$

$$2266711743\lambda^4 - 28544560\lambda^2 - 2612639\lambda + 185279 = 0; \quad M = 69 \quad \text{(A69)}$$

$$2400999638\lambda^4 - 29378286\lambda^2 - 2650282\lambda + 188564 = 0; \quad M = 70 \quad \text{(A70)}$$

$$2541167734\lambda^4 - 30224012\lambda^2 - 2687930\lambda + 191859 = 0; \quad M = 71 \quad \text{(A71)}$$

$$2687385229\lambda^4 - 31081736\lambda^2 - 2725582\lambda + 195164 = 0; \quad M = 72 \quad \text{(A72)}$$

$$2839823724\lambda^4 - 31951460\lambda^2 - 2763240\lambda + 198479 = 0; \quad M = 73 \quad \text{(A73)}$$

$$2998657220\lambda^4 - 32833183\lambda^2 - 2800902\lambda + 201803 = 0; \quad M = 74 \quad \text{(A74)}$$

$$3164062115\lambda^4 - 33726905\lambda^2 - 2838568\lambda + 205137 = 0; \quad M = 75 \quad \text{(A75)}$$

$$3336217210\lambda^4 - 34632627\lambda^2 - 2876239\lambda + 208479 = 0; \quad M = 76 \quad \text{(A76)}$$

$$3515303706\lambda^4 - 35550348\lambda^2 - 2913915\lambda + 211830 = 0; \quad M = 77 \quad \text{(A77)}$$

$$3701505201\lambda^4 - 36480068\lambda^2 - 2951594\lambda + 215189 = 0; \quad M = 78 \quad \text{(A78)}$$

$$3895007696\lambda^4 - 37421787\lambda^2 - 2989278\lambda + 218556 = 0; \quad M = 79 \quad \text{(A79)}$$

$$4095999592\lambda^4 - 38375506\lambda^2 - 3026965\lambda + 221931 = 0; \quad M = 80 \quad \text{(A80)}$$

$$4304671687\lambda^4 - 39341224\lambda^2 - 3064657\lambda + 225313 = 0; \quad M = 81 \quad \text{(A81)}$$

$$4521217182\lambda^4 - 40318941\lambda^2 - 3102352\lambda + 228703 = 0; \quad M = 82 \quad \text{(A82)}$$

$$4745831678\lambda^4 - 41308658\lambda^2 - 3140050\lambda + 232100 = 0; \quad M = 83 \quad (A83)$$

$$4978713173\lambda^4 - 42310375\lambda^2 - 3177752\lambda + 225504 = 0; \quad M = 84 \quad (A84)$$

$$5220062068\lambda^4 - 43324090\lambda^2 - 3215458\lambda + 238915 = 0; \quad M = 85 \quad (A85)$$

$$5470081163\lambda^4 - 44349806\lambda^2 - 3253166\lambda + 242333 = 0; \quad M = 86 \quad (A86)$$

$$5728975659\lambda^4 - 45387520\lambda^2 - 3290878\lambda + 245756 = 0; \quad M = 87 \quad (A87)$$

$$5996953154\lambda^4 - 46437234\lambda^2 - 3328593\lambda + 249186 = 0; \quad M = 88 \quad (A88)$$

$$6274223649\lambda^4 - 47498948\lambda^2 - 3366311\lambda + 252622 = 0; \quad M = 89 \quad (A89)$$

$$6560999544\lambda^4 - 48572661\lambda^2 - 3404032\lambda + 256064 = 0; \quad M = 90 \quad (A90)$$

$$6857495640\lambda^4 - 49658374\lambda^2 - 3441755\lambda + 259511 = 0; \quad M = 91 \quad (A91)$$

$$7163929135\lambda^4 - 50756086\lambda^2 - 3479482\lambda + 262963 = 0; \quad M = 92 \quad (A92)$$

$$7480519630\lambda^4 - 51865798\lambda^2 - 3517211\lambda + 266421 = 0; \quad M = 93 \quad (A93)$$

$$7807489125\lambda^4 - 52987510\lambda^2 - 3554942\lambda + 269884 = 0; \quad M = 94 \quad (A94)$$

$$81450620\lambda^4 - 54121221\lambda^2 - 3592676\lambda + 273352 = 0; \quad M = 95 \quad (A95)$$

$$8493465116\lambda^4 - 55266931\lambda^2 - 3630412\lambda + 276824 = 0; \quad M = 96 \quad (A96)$$

$$8852927611\lambda^4 - 56424642\lambda^2 - 3668151\lambda + 280301 = 0; \quad M = 97 \quad (A97)$$

$$9223681106\lambda^4 - 57594351\lambda^2 - 3705891\lambda + 283783 = 0; \quad M = 98 \quad (A98)$$

$$9605959601\lambda^4 - 58776061\lambda^2 - 3743634\lambda + 287269 = 0; \quad M = 99 \quad (A99)$$

$$9999999496\lambda^4 - 59969770\lambda^2 - 3781379\lambda + 290759 = 0; \quad M = 100 \quad (A100)$$

The limiting value of the force of interest is given by

$$\frac{i^{(m)}}{m} = \left[(1+i)^{\frac{1}{m}} - 1 \right] \quad (A101)$$

APPENDIX B

Presentation of Results

Table 1: Interest Rate Intensity from Model Equations

$ROOT\ 1$	$ROOT\ 2$	$ROOT\ 3$	$ROOT\ 4$	$LIMIT$
-25.3601	12.6557+20.5362000i	12.6557-20.5362000i	0.0487902	0.050000
5.11237	-2.58058+2.8228900i	-2.58058-2.8228900i	0.048791	0.0493902
3.36729	-1.70804+1.6299500i	-1.70804-1.6299500i	0.0487897	0.0491891
2.51479	-1.28179+1.1424400i	-1.28179-1.1424400i	0.048788	0.0490889
2.00506	-1.02693+0.8789500i	-1.02693-0.8789500i	0.0487906	0.049029
1.66599	-0.85739+0.7146060i	-0.85739-0.7146060i	0.0487918	0.0489891
1.42427	-0.73653+0.6025580i	-0.73653-0.6025580i	0.0487895	0.0489606
1.24331	-0.646051+0.521380i	-0.646051-0.521380i	0.0487908	0.0489392
1.1028	-0.575795+0.459905i	-0.575795-0.459905i	0.0487897	0.0489227
0.990558	-0.519674+0.411761i	-0.519674-0.411761i	0.0487896	0.0489094
0.898841	-0.473816+0.373047i	-0.473816-0.373047i	0.0487903	0.0488985
0.822497	-0.435643+0.341247i	0.435643-0.3412470i	0.04879	0.0488895
0.757959	-0.403375+0.314665i	-0.403375-0.314665i	0.0487911	0.0488818
0.702688	-0.375739+0.292115i	-0.375739-0.292115i	0.0487903	0.0488753
0.65482	-0.351806+0.272747i	-0.351806-0.272747i	0.0487911	0.0488696
0.612961	-0.330876+0.255932i	-0.330876-0.255932i	0.048791	0.0488646
0.576045	-0.312418+0.241197i	-0.312418-0.241197i	0.0487914	0.0488602
0.543243	-0.296017+0.228178i	-0.296017-0.228178i	0.0487915	0.0488563
0.513904	-0.281348+0.216593i	-0.281348-0.216593i	0.0487915	0.0488529
0.487506	-0.268149+0.206216i	-0.268149-0.206216i	0.0487918	0.0488497
0.463625	-0.256209+0.196869i	-0.256209-0.196869i	0.0487926	0.0488469
0.441919	-0.245356+0.188404i	-0.245356-0.188404i	0.048793	0.0488443
0.4221	-0.235447+0.180703i	-0.235447-0.180703i	0.0487944	0.048842
0.403933	-0.226364+0.173666i;	-0.226364-0.173666i	0.0487951	0.0488398
0.387219	-0.218007+0.167210i	-0.218007-0.167210i	0.0487956	0.0488378
0.371789	-0.210293+0.161266i	-0.210293-0.161266i	0.0487968	0.048836
0.357499	-0.203149+0.155776i	-0.203149-0.155776i	0.0487982	0.0488343
0.344228	-0.196514+0.150690i	-0.196514-0.150690i	0.0488	0.0488327
0.331868	-0.190335+0.145963i	-0.190335-0.145963i	0.0488017	0.0488312
0.320329	-0.184567+0.141559i	-0.184567-0.141559i	0.0488037	0.0488299
0.309531	-0.179169+0.137446i	-0.179169-0.137446i	0.048806	0.0488286
0.299404	-0.174106+0.133595i	-0.174106-0.133595i	0.0488084	0.0488274
0.289887	-0.169349+0.129983i	-0.169349-0.129983i	0.0488111	0.0488262
0.280926	-0.16487+0.1265870i	-0.16487-0.1265870i	0.0488142	0.0488252
0.272472	-0.160645+0.123388i	-0.160645-0.123388i	0.0488181	0.0488242
0.264483	-0.156652+0.120370i	-0.156652-0.120370i	0.0488217	0.0488232
0.256922	-0.152874+0.117517i	-0.152874-0.117517i	0.0488259	0.0488223
0.249755	-0.149293+0.114816i	-0.149293-0.114816i	0.0488309	0.0488215
0.24295	-0.145893+0.112255i	-0.145893-0.112255i	0.0488358	0.0488207
0.236481	-0.142661+0.109824i	-0.142661-0.109824i	0.0488414	0.0488199
0.230323	-0.139585+0.107512i	-0.139585-0.107512i	0.0488475	0.0488192
0.224454	-0.136654+0.105311i	-0.136654-0.105311i	0.0488545	0.0488185

0.218853	-0.133857+0.103213i	-0.133857-0.103213i	0.0488615	0.0488179
0.213502	-0.131186+0.101210i	-0.131186-0.101210i	0.0488695	0.0488172
0.208384	-0.128631+0.0992967i	-0.128631-0.0992967i	0.0488781	0.0488166
0.203484	-0.126186+0.0974665i	-0.126186-0.0974665i	0.0488874	0.048816
0.198788	-0.123842+0.0957141i	-0.123842-0.0957141i	0.0488973	0.0488155
0.194282	-0.121595+0.0940346i	-0.121595-0.0940346i	0.0489081	0.048815
0.189956	-0.119438+0.0924233i	-0.119438-0.0924233i	0.04892	0.0488145
0.185798	-0.117365+0.0908760i	-0.117365-0.0908760i	0.0489324	0.048814
0.181798	-0.115372+0.0893890i	-0.115372-0.0893890i	0.0489458	0.0488135
-0.113454+0.0879587i	-0.113454-0.08795870i	0.177947	0.0489604	0.0488131
-0.111606+0.0865817i	-0.111606-0.08658170i	0.174237	0.0489757	0.0488126
-0.109826+0.0852551i	-0.109826-0.0852551i	0.170659	0.0489923	0.0488122
-0.108108+0.0839761i	-0.108108-0.0839761i	0.167206	0.0490098	0.0488118
-0.10645+0.08274200i	-0.10645-0.08274200i	0.163872	0.0490286	0.0488114
-0.104849+0.0815505i	-0.104849-0.0815505i	0.16065	0.0490485	0.0488111
-0.103302+0.0803993i	-0.103302-0.0803993i	0.157534	0.04907	0.0488107
-0.101805+0.0792864i	-0.101805-0.0792864i	0.154518	0.0490928	0.0488103
-0.100357+0.0782097i	-0.100357-0.0782097i	0.151598	0.0491169	0.04881
-.0989553+0.0771675i	-0.0989553-0.0771675i	0.148768	0.0491425	0.0488097
-0.097597+0.07615800i	-0.097597-0.07615800i	0.146024	0.0491697	0.0488094
-0.0962806+0.0751798i	-0.0962806-0.0751798i	0.143362	0.0491987	0.0488091
-0.0950038+0.0742313i	-0.0950038-0.0742313i	0.140778	0.0492294	0.0488088
-0.0937651+0.0733111i	-0.0937651-0.0733111i	0.138268	0.0492619	0.0488085
-0.0925625+0.0724179i	-0.0925625-0.0724179i	0.135829	0.0492965	0.0488082
-0.0913946+0.0715505i	-0.0913946-0.0715505i	0.133456	0.049333	0.0488079
-0.0902597+0.0707077i	-0.0902597-0.0707077i	0.131148	0.0493717	0.0488077
-0.0891565+0.0698885i	-0.0891565-0.0698885i	0.1289	0.0494126	0.0488074
-0.0880836+0.0690918i	-0.0880836-0.0690918i	0.126711	0.0494558	0.0488072
-0.0870396+0.0683166i	-0.0870396-0.0683166i	0.124578	0.0495015	0.0488069
-0.0860235+0.0675621i	-0.0860235-0.0675621i	0.122497	0.0495498	0.0488067
-0.0850341+0.0668275i	-0.0850341-0.0668275i	0.120467	0.0496009	0.0488065
-0.0840703+0.0661118i	-0.0840703-0.0661118i	0.118486	0.0496549	0.0488063
-0.0831312+0.0654143i	-0.0831312-0.0654143i	0.11655	0.0497122	0.048806
-0.0822156+0.0647343i	-0.0822156-0.0647343i	0.114659	0.0497726	0.0488058
-0.0813227+0.0640712i	-0.0813227-0.0640712i	0.112809	0.0498365	0.0488056
-0.0804516+0.0634242i	-0.0804516-0.0634242i	0.110999	0.0499039	0.0488054
-0.0796016+0.0627928i	-0.0796016-0.0627928i	0.109228	0.0499752	0.0488052
-0.0787718+0.0621763i	-0.0787718-0.0621763i	0.107493	0.0500507	0.048805
-0.0779615+0.0615742i	-0.0779615-0.0615742i	0.105793	0.0501303	0.0488049
-0.07717+0.060986000i	-0.07717-0.060986000i	0.104125	0.0502148	0.0488047

-0.0763966+0.0604111i	-0.0763966-0.0604111i	0.102489	0.0503042	0.0488045
-0.0756407+0.0598492i	-0.0756407-0.0598492i	0.100882	0.050399	0.0488043
-0.0749017+0.0592997i	-0.0749017-0.0592997i	0.0993039	0.0504994	0.0488042
-0.074179+0.05876230i	-0.074179-0.05876230i	0.0977518	0.0506061	0.048804
-0.0734719+0.0582364i	-0.0734719-0.0582364i	0.0962246	0.0507192	0.0488038
-0.0727801+0.0577217i	-0.0727801-0.0577217i	0.0947206	0.0508395	0.0488037
-0.0721029+0.0572178i	-0.0721029-0.0572178i	0.0932383	0.0509676	0.0488035
-0.07144+0.056724500i	-0.07144-0.056724500i	0.0917759	0.0511041	0.0488034
-0.0707909+0.0562412i	-0.0707909-0.0562412i	0.0903319	0.0512498	0.0488032
-0.070155+0.05576770i	-0.070155-0.05576770i	0.0889046	0.0514054	0.0488031
-0.069532+0.05530370i	-0.069532-0.05530370i	0.0874917	0.0515723	0.048803
-0.0689215+0.0548489i	-0.0689215-0.0548489i	0.0860915	0.0517516	0.0488028
-0.0683231+0.0544030i	-0.0683231-0.0544030i	0.0847017	0.0519446	0.0488027
-0.0677364+0.0539657i	-0.0677364-0.0539657i	0.0833199	0.0521529	0.0488026
-0.0671611+0.0535367i	-0.0671611-0.0535367i	0.0819433	0.0523789	0.0488024
-0.0665968+0.0531159i	-0.0665968-0.0531159i	0.0805685	0.0526251	0.0488023
-0.0660432+0.0527030i	-0.0660432-0.0527030i	0.079192	0.0528945	0.0488022
-0.0655+0.0522976000i	-0.0655-0.0522976000i	0.0778089	0.0531912	0.0488021

Appendix C

The Trajectories of Interest Rate Intensity and its Limiting Value

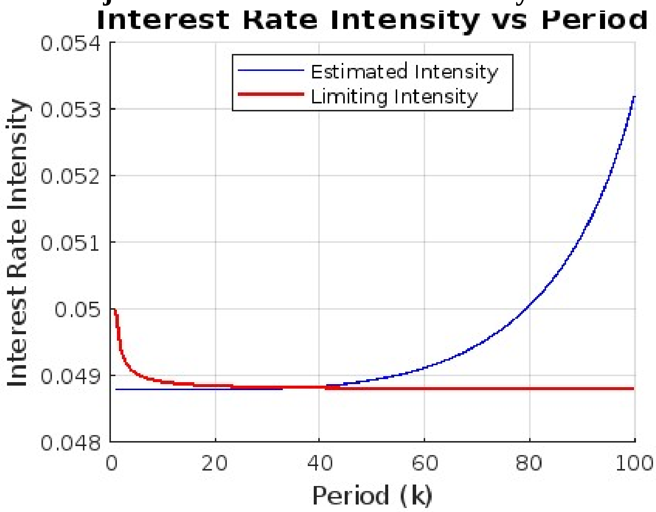

Figure 1: Comparison of Estimated intensity and limiting value