Two-particle Quantum Walks over a Line

Anuradha Mahasinghe[1], Jingbo Wang[2], and Jagath Wijerathna[1]

[1]*Departmet of Mathematics, Faculty of Science, University of Colombo*

[2]*School of Physics, University of Western Australia*

Introduction

Quantum computing, the quantum analogue of classical computing makes use of qubits - quantum analogue of classical bits, as the elementary quantum registers of storing, manipulating and measuring data (Nakahara and Ohmi, 2008). Mathematically, a qubit is a unit vector of the form $c_1|0\rangle + c_2|1\rangle$ so that $|c_1|^2 + |c_2|^2 = 1$ in a Hilbert space spanned by canonical basis states $|0\rangle = \begin{bmatrix}1\\0\end{bmatrix}$ and $|1\rangle = \begin{bmatrix}0\\1\end{bmatrix}$. When a qubit is queried, obtained is a probabilistic answer, instead of a deterministic one. That will be, the state $|0\rangle$ with probability $|c_1|^2$, and $|1\rangle$ with probability$|c_2|^2$. Also quantum algorithms were designed following this quantization. Various strategies for quantum algorithms emerged, along with the one based upon the idea introduced by Aharanov [4], and it was developed under the term 'quantum walks'. Currently, two main categories of quantum walks are being considered: discrete and continuous quantum walks; while discrete quantum walks were studied under two subcategories as Markov chain-based and coin-based walks.

Coin-based quantum walk is the explicit quantum analogue of classical random walk, in which the walker is at some position on a discretized straight line initially, where the coin flip takes place to decide either to move to left or to right. The walk is thus determined by the probability of finding the walker at some point at a given time. Mathematically, the state of the walker is thus the tensor product of coin states and the vertex states (.Berry and Wang, 2010). The most accepted mathematical model states that the walker is in the Hilbert space $\mathcal{H}_c \otimes \mathcal{H}_P$ where $\mathcal{H}_c$ is the coin state, spanned by the orthonormal basis states $|1\rangle_C$ and $|0\rangle_C$, and $\mathcal{H}_P$ is the position Hilbert space spanned by the basis states of vertices on the straight line the walker moves. Thus, a generic state of the walker can be expressed as $|\psi(t)\rangle = \sum_{n=-\infty}^{\infty} \alpha_L(n,t)|1\rangle_C|n\rangle_p + \alpha_R(n,t)|o\rangle_C|n\rangle_p \in \mathcal{H}_c \otimes \mathcal{H}_P$, where $\alpha_L(n,t)$ and $\alpha_R(n,t)$ are left and right amplitudes respectively, of the walker at position n, time t.

Hadamard Walk over a Line

Outcome of the coin flip in a quantum walk depends upon the choice of the coin. Commonly used coins are Grover's coin, Hadamard coin, balanced coin, Pauli's coin etc. For instance, if Hadamard coin $\widehat{H}$ was selected, the evolution of the walk could be regarded as repeated alternative application of $\widehat{H} \otimes \hat{I}$ and the shift $\hat{T}$ defined as follows, with the initial state $|0\rangle_C|0\rangle_p$: The coin operation is defined as $\widehat{H}$: $\mathcal{H}_c \rightarrow \mathcal{H}_c$ so that $c_1|0\rangle_C + c_2|1\rangle_C \xrightarrow{\widehat{H}} \frac{1}{\sqrt{2}}(c_1 + c_2)|0\rangle + \frac{1}{\sqrt{2}}(c_1 - c_2)|1\rangle$, and the shift operation is defined as $\hat{T}: \mathcal{H}_c \otimes \mathcal{H}_P \rightarrow \mathcal{H}_c \otimes \mathcal{H}_P$ so that $|0\rangle_C|n\rangle_p \xrightarrow{\hat{T}} |0\rangle_C|n+1\rangle_p$, $|1\rangle_C|n\rangle_p \xrightarrow{\hat{T}} |1\rangle_C|n-1\rangle_p$.

Analytical Solution of Hadamard Walk

An analytical solution of the Hadamard walk along the straight line was derived by Nayak and Vishwanath (2008). The methodology adopted by them was reducing the time evolution of the system into the eigenvalue problem in the Fourier space by discrete Fourier transformation. First, the state of the walker at the position n at time $t+1$ is expressed in terms of the states at the positions $n-1$ and $n+1$ at time t by $|\psi(n,t+1)\rangle = M_+ \, |\psi(n-1,t)\rangle + M_-|\psi(n+1,t)\rangle$, where $M_+ = \begin{bmatrix} 0 & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$ and $M_- = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 \end{bmatrix}$. Then in the Fourier space it takes the form $|\bar{\psi}(k,t+1)\rangle = \overline{M_k}|\bar{\psi}(k,t)\rangle$ for $\overline{M_k} = e^{ik}M_+ + e^{-ik}M_-$, and after solving with the relevant initial condition, and transformed back to the time domain, the solution was obtained (Aharanov *et al*, 1993). The importance of this solution is, its extendibility for regular graphs, as quantum walk on the infinite straight line was found to be isomorphic to the quantum walk on certain regular graphs, as shown by N. Shenvi, J. Kempe and K. Whaley(2003).

Two-particle Quantum Walks

For many potential practical applications of quantum walks, multi-particle walkers are preferred instead of single particle walkers. A two particle walker has several possibilities to interact: they might evolve their walks individually, independent of the states of each other, or they can interact when confronted Berry and Wang, 2011). Needless to say, the walk takes place in the Hilbert space$(\mathcal{H}_c \otimes \mathcal{H}_P) \otimes (\mathcal{H}_c \otimes \mathcal{H}_P)$. Identity interaction is one way of interacting, in which the coin operator $\hat{C}$ collapses down to the identity operator at common vertices. For the two-particle Hadamard walk, $\hat{C} = \hat{H} \otimes \hat{H}$ when the two particles are at different vertices, and $\hat{C} = \hat{I}$ when they are at the same vertex. In π-phase interaction, another possibility of interacting, rather than using the identity coin operator, the phase of all vertex states are being shifted by π. However, the non-interacting quantum particles are also not totally independent of each other due to quantum entanglement. Regarding the identity interaction, when entanglement is neglected, the problem can be transformed to the following form which might be helpful in obtaining analytical solutions:

If the 4×4 matrix P_i is defined for each $i = 1,2,3,4$ as $\mathrm{P}_i = (p_{jk})$ so that $p_{jk} = \begin{cases} 1, & j = k = i \\ 0, & \text{otherwise} \end{cases}$, and

$$N_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 1/2 & 1/2 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$N_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/2 & -1/2 & 1/2 & -1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, N_4 = \begin{bmatrix} 1/2 & -1/2 & -1/2 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

then the state at positions n, m at time $t+1$ is given by

$|\psi((n,t+1),(m,t+1))\rangle = M_1|\psi((n-1,t),(m-1,t))\rangle + M_2|\psi((n-1,t),(m+1,t))\rangle + M_3|\psi((n-1,t),(m+1,t))\rangle + M_4|\psi((n+1,t),(m+1,t))\rangle$,

where,

$M_i = N_i$for $m \neq n$ and $m \neq n \pm 2$, $M_1 = N_1$, $M_2 = P_2$, $M_3 = N_3$ and $M_4 = N_4$,for $m = n + 2$, $M_1 = N_1$, $M_2 = N_2$, $M_3 = P_3$ and $M_4 = N_4$, for $m = n - 2$, $M_1 = P_1$, $M_2 = N_2$, $M_3 = N_3$ and $M_4 = P_4$.for $m = n$,

References

Aharanov, Y., Davidovich, L. and Zagury, N. (1993). Quantum Random Walks. Physical Review A , 48, 2

Berry, S. D. and Wang, J. B. (2010). Quantum-walk-based Search and Centrality. Phys. Rev. 0.1103/ Phys. Rev. 82 042333.

Berry, S. D. and Wang, J. B. (2011). Two-particle Quantum Walks: Entanglement and Graph Isomorphism Testing. Physical Review A , Volume 83

Nakahara, M and Ohmi, T. (2008) Quantum Computing: From Linear Algebra to Physical Realizations. CRC Press, Taylor and Francis Group, Boca Raton.

Shenvi,N., Kempe J. and Whaley, K. (2003). Quantum Random-walk Search Algorithm. Physical Review A 67

Venegas-Andraca, S.E.. (2008).Quantum Walks for Computer Scientists. Morgan and Claypool Publishers,