Normal Multiresolution Approximation of Piecewise Smooth Images

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Supervisor 1: Prof. N.D. Kodikara Date:..................

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Abstract

This dissertation sets a novel approach to analyze second generation wavelet schemes by providing a basis function and decomposition method. Moreover, the representation of gray–scale images with normal multiresolution approximation in less smooth spaces, such as Besov spaces, $B^{\alpha}_{p,q}(\overline{\Omega})$, $1 \leq p, q \leq \infty$ for $0 < \alpha < 1$, where Ω be a Lipschitz domain in $\mathbb{R}^d$, $d \geq 1$, and d is odd.

The assumption is that a normal multiresolution approximation is parameterized in a regular interval, and then with the Lagrangian interpolation formula a basis function is constructed by using Hardy's multiquadric function. The basis function is shift–invariant and, generates a space $\mathcal{S}_j = span\{\varphi(2^j x - k) : \ for \ all \ x \in \mathbb{R} \ and \ for \ all \ k \in \mathbb{Z}\}$ for $j \in \mathbb{N}_0$. Approximation properties of this setting is explored in Sobolev spaces.

Since the above basis function does not satisfy the requirements of a compact support; it is resort to consider the second divided difference of the basis function. Thus, the wavelet transform on the real line is defined on the basis of quasi–interpolating basis function. In addition, the local properties of the function are also studied; for instance, the case of pointwise convergence.

As such, the above stated basis function is generalized to multivariate setting in a bounded simply connected domain $\Omega \subset \mathbb{R}^d$, $d \geq 1$, with the localization concept of multiquadric functions and 1–unisolvence property. Thus, the characterization of the Besov spaces, $B^{\alpha}_{p,q}(\Omega)$, in terms of vertical offset coefficients of functions with respect to these bases. As a consequence, it is seen that Horizon images with $0 < \alpha < 1$ are characterized by the coefficients with respect to these normal wavelet basis functions.

As an application of the multiquadric basis function, an efficient image compression scheme, called Normal Multiresolution Triangulation Interpolation scheme, is presented in this dissertation.

Contents

List of Figures

Notations and Inequalities

We shall use the following standard notation:

$\mathbb{N}$ – the set of all natural numbers,

$\mathbb{N}_0$ – the set of all nonnegative integers,

$\mathbb{Z}$ – the set of all integers,

$\mathbb{R}$ – the set of all real numbers,

$\mathbb{N}_0^n = \underbrace{\mathbb{N}_0 \times \cdots \times \mathbb{N}_0}_{n}$ – the set of multi–indices (n is the natural number which will be used exclusively to denote the dimension),

$\mathbb{R}^n = \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{n}$,

$B(x, r)$ – the open ball of radius $r > 0$ centered at the point $x \in \mathbb{R}^n$,

Ω^c $(\Omega \subset \mathbb{R}^n)$ – the complement of Ω in $\mathbb{R}^n$,

$\overline{\Omega}$ $(\Omega \subset \mathbb{R}^n)$ – the closure of Ω,

$\underline{\Omega}$ $(\Omega \subset \mathbb{R}^n)$ – the interior of Ω.

For an arbitrary nonempty set $\Omega \subset \mathbb{R}^n$ we shall denote by:

$C(\Omega)$– the space of functions continuous on Ω,

$C_b(\Omega)$– the Banach space of functions continuous and bounded on Ω with the norm

$$\|f\| = \sup_{x \in \Omega} |f(x)|.$$

For a measurable nonempty set $\Omega \subset \mathbb{R}^n$ we shall denote by:

$L_p(\Omega)$ $(1 \le p < \infty)$– the Banach space of functions f measurable on Ω such that the norm

$$\|f\|_{L_p(\Omega)} = \left(\int_\Omega |f|^p dx \right)^{\frac{1}{p}} < \infty,$$

$L_\infty(\Omega)$ – the Banach space of functions f measurable on Ω such that the norm

$$\|f\|_{L_\infty(\Omega)} = ess \sup_{x \in \Omega} |f(x)| = \inf_{\omega : meas\omega = 0} \sup_{x \in \Omega \setminus \omega} |f(x)| < \infty.$$

By $\|f\|_{L_p(\Omega)}$, $(1 \le p \le \infty)$ we denote the linear space of all functions f on Ω such that $\|f\|_{L_p(\Omega)} < \infty$. Equipped with the norm $\| \cdot \|_{L_p(\Omega)}$, $L_p(\Omega)$, becomes a Banach space. For $p = 2$, $L_2(\Omega)$ is a

Hilbert space with the inner product given by

$$\langle f,g\rangle = \int_\Omega f(x)g(x)dx, \quad f,g\in L_2(\Omega)$$

For an open nonempty set $\Omega\subset\mathbb{R}^n$ we shall denote by:

$L_p^{loc}(\Omega)$ $(1\le p\le\infty)$– the set of functions defined on Ω such that for each compact $K\subset\Omega$, $f\in L_p(K)$,

$C^\infty(\Omega)=\bigcap_\ell^\infty C^\ell(\Omega)$– the space of infinitely continuously differentiable functions on Ω,

$C_0^\infty(\Omega)$ – the space of functions in $C^\infty(\Omega)$ compactly supported,

$W_p^\ell(\Omega)$ $(\ell\in\mathbb{N},\ 1\le p\le\infty)$– Sobolev space, which is the Banach space of functions $f\in L_p(\Omega)$ such that for all $\alpha\in\mathbb{N}_0^n$ where $|\alpha|=\ell$ the weak derivatives $D_w^\alpha f$ exist on Ω and $D_w^\alpha f\in L_p(\Omega)$ with the norm

$$\|f\|_{W_p^\ell(\Omega)}=\|f\|_{L_p(\Omega)}+\sum_{|\alpha|=\ell}\|D_w^\alpha f\|_{L_p(\Omega)L_p(\Omega)},$$

$w_p^\ell(\Omega)$ $(\ell\in\mathbb{N},\ 1\le p\le\infty)$– the semi–normed Sobolev space, which is the semi–Banach space of functions $f\in L_1^{loc}(\Omega)$ such that for all $\alpha\in\mathbb{N}_0^n$ where $|\alpha|=\ell$ the weak derivatives $D_w^\alpha f$ exist on Ω and $D_w^\alpha f\in L_p(\Omega)$ with the semi–norm

$$\|f\|_{w_p^\ell(\Omega)}=\sum_{|\alpha|=\ell}\|D_w^\alpha f\|_{L_p(\Omega)}.$$

Let k be a nonnegative integer and $C^k(\mathbb{R}^n)$ be the space of continuous functions in $\mathbb{R}^n$ having continuous partial derivatives up to order k.

In $C^k(\mathbb{R}^n)$ the usual topology defined by the family of semi–norms:

$$|f;C^k(\mathbb{R}^n)|=\max_{|j|\le k}\max_K|D_jf|,$$

where $K\subset\mathbb{R}^n$ are compact.

For $k<\alpha\le k+1$, $Lip(\alpha,\mathbb{R}^n)$ the Lipschitz space of order α, corresponding to first difference, is the space of functions $f\in C^k(\mathbb{R}^n)$ such that

$$|D_jf(x)|\le M \quad for\ |j|\le k\ and\ x\in\mathbb{R}^n$$

and

$$|\Delta_hD_jf(x)|\le M|h|^{\alpha-k} \quad for\ |j|=k\ and\ x,h\in\mathbb{R}^n,$$

where $\Delta_hg(x)=g(x+h)-g(x)$ and the norm $\|f;Lip(\alpha,\mathbb{R}^n)\|$ is the infimum of all possible constants M in these inequalities. The space $\Lambda_\alpha(\mathbb{R}^n)$ is the Lipschitz space of order α corresponding to second differences.

Recall that Lip_M1 is the set of all continuous functions f such that

$$\|Df(x)\|_\infty\le M \quad for\ all\ x\in\mathbb{R}.$$

Hölder's inequality: Suppose that $\frac{1}{p} + \frac{1}{q} = 1$, for $1 < p < \infty$, $q = \infty$ for $p = 1$ and $q = 1$ for $p = \infty$. If $f \in L_p(\Omega)$ and $g \in L_q(\Omega)$ then $fg \in L_1(\Omega)$ and $\|fg\|_{L_1(\Omega)} \leq \|f\|_{L_p(\Omega)}\|g\|_{L_q(\Omega)}$.
Minkowski's inequality. If $f, g \in L_p(\Omega)$ then $f + g \in L_p(\Omega)$ and $\|f + g\|_{L_p} \leq \|f\|_{L_p} + \|g\|_{L_p}$.
Minkowskis inequality for integrals. In addition, let $A \subset \mathbb{R}^n$ be a measurable set. Suppose that f is measurable on $A \times \Omega$ and $f(, y) \in L_p(\Omega)$ for almost all $y \in A$. Then

$$\| \int_A f(, y)dy\|_{L_p(\Omega)} \leq \int_A \|f(, y)dy\|_{L_p(\Omega)}dy,$$

if the right-hand side is finite. Similar inequalities hold for finite and infinite sums.
The Fourier transform of $f \in L_1(\mathbb{R}^s)$ is defined as

$$\hat{f}(\theta) = \int_{\mathbb{R}^s} f(t)exp(-i\theta \cdot t)dt$$

The Fourier transform can be uniquely extended to functions in $L_2(\mathbb{R}^s)$. For $\mu > 0$ we denote by H^μ the space of all functions $f \in L_2(\mathbb{R}^s)$ such that the semi–norm

$$|f|_{H^\mu(\mathbb{R}^s)} := \left(\int_{\mathbb{R}^s} |\hat{f}(\xi)|^2 |\xi|^{2\mu} d\xi \right)^{1/2}$$

is finite. For a nonempty open subset Ω of $\mathbb{R}^s$, we define

$$H^\mu(\Omega) := \{f|_\Omega : f \in H^\mu(\mathbb{R}^s)\}.$$

Chapter 1

Introduction

Nowadays surfaces in Computer Aided Geometric Design are often described with millions of control parameters. Guskov et al. (2000) introduced normal multiresolution approximation of curves or surfaces. Subdivision scheme can be combined with this approximation scheme to approximate arbitrary functions, curves and surfaces. A multiresolution approximation of a curve or surface is normal if all the wavelet vectors perfectly align with a locally defined normal direction which only depends on the coarser levels. Note that by the normal direction we mean a normal onto an approximation of the curve or surface. Normal multiresolution approximation depends on the computation of a normal, as such this approximation leads to nonlinear representation of wavelet coefficients. This scheme is known as normal multiresolution approximation and the detail vectors in normal directions are called normal wavelets.

The mathematical properties of these wavelets are well understood as the approximation of functions of one or more variables with some regularity conditions imposed on the functions f, such as the continuity of the function being interpolated. Daubechies et al. (2004) investigated the mathematical properties of normal multiresolution approximation of curves, such as convergence, regularity and stability of smooth curves and it is shown numerically that normal meshes are stable. However, for the case of 1 D curves in the plane or 2 D surfaces in 3 D spaces much less is known about their convergence and stabilities in less smooth spaces, such as Besov spaces. These are the essential properties to be studied in the context of surface compression.

Surface compression, which is in fact a trade off between maintaining accuracy and reduction of the amount of data, is essential in these contexts. As this normal direction depends on coarser level, only a single scaler coefficient need to be stored instead of 2 or 3–vectors. Typically one takes a parametrization of the original curve or surface and ends up with wavelet analysis in each of the two or three components. This means the wavelet coefficients now become 2 or 3–vectors. This is useful in compression of curves or surfaces.

A surface compression algorithm is given in DeVore et al. (1992b) by means of wavelet decomposition of certain box splines, and error bounds are given in terms of the input surfaces. However, multiresolution triangulation are widely used in computer graphics for representing 3 D shapes. Also, it is useful to describe 2 D surfaces, gray–scale images in some function classes, such as Horizon class images (Donoho, 1999) which comprise constant regions separated by smooth discontinuities, where the line of discontinuity is Hölder's continuous. Normal multiresolution triangulation is an efficient triangulation to the local adaptivity and to the discontinuities. Normal meshes automatically generate a polyline (piecewise linear) approximation which gives us optimal rate of error decay in certain function classes unlike the blocky piecewise constant approximation of tensor product wavelets. In this way, the proposed nonlinear multiscale normal mesh decomposition is an anisotropic representation of the 2 D function. The same idea of anisotropic representations lies at the basis of decompositions, such as wedgelet and curvelet transforms but, the proposed normal mesh approach has a unique construction.

The purpose of this research is to extend these ideas to the case of multiresolution analysis over normal multiresolution triangle. The challenging task is to describe the quasi–interpolating wavelet functions over the normal multiresolution triangles and efficient representation (compressed) of gray–scale images with various smoothness properties.

1.1 Background and Motivation

For the scalar algorithm on a uniform grid, the interpolating process is equivalent to a scheme proposed by Dubuc (1986) and later extended by Deslauriers and Dubuc (1989). Dubuc defines a method of interpolation generated through a symmetric iterative process on the refined dyadic grids. The Dubuc iterative process is an interpolation scheme, i.e., a fill–in scheme, but not a multiresolution scheme. Donoho (1992) developed an interpolating wavelet transform using Deslauriers and Dubuc interpolating function. This is a non–orthogonal transform with formal resemblance to orthogonal wavelet transform (Cohen et al., 1992, 1993a,b; Daubechies, 1988). It represents the objects by dilation and translation of a basis function, but for which the coefficients are obtained from linear combination of samples rather than integrals. This transform depends in a fundamental way on the interpolation scheme of Deslauriers and Dubuc (1989). An important observation of this transformation is that the resulting coefficients had the decay properties as the decay properties of smooth orthogonal wavelet decomposition. It is essentially C^2 smoothness (Daubechies, 1992, Chap. 6) interpolating wavelet transform.

A quasi–interpolatory multiresolution algorithm of Guskov et al. (2000) is known as normal multiresolution approximation. This scheme is similar to the lifting scheme proposed by Sweldens (1996). The scalar version of the Harten's (1993) (Abgrall and Harten, 1998; Harten, 1996) algorithm is closely related to the work of Donoho (1992) and Sweldens (1997). The essential fact in this schemes is quasi–interpolating multiresolution algorithm.

1.1.1 Interpolating Wavelet Transforms

There are three well known types of constructions for interpolating wavelet transforms. The first one is that interpolating spline wavelets. Let D be an odd positive integer, and L_D be the fundamental polynomial spline of degree D (Schoenberg, 1972), i.e., the piecewise polynomial with knots at the integers $k \in \mathbb{Z}$, continuity C^{D-1}, and satisfying the interpolation conditions. This function is a $(D-1, D)$ interpolating wavelet; it is regular of order $R = D - 1$; its derivatives through order $D - 1$ decay exponentially with distance from 0; it satisfies a two–scale relation; and it generates all polynomials of degree

D through its translates.

The second family is that the Deslauriers–Dubuc Fundamental functions. Let D be an odd positive integer. These are functions F_D defined by interpolating the Kronecker sequence at the integers on a function on the binary rationals by repeated application of the following rule. If F_D has already been defined at all binary rationals with denominator 2^j, $j \geq 0$, extend it by polynomial interpolation to all binary rationals with denominator 2^{j+1}, i.e., all points halfway between previously defined points. Specifically, to define the function at $(k + 1/2)/2^j$ when it is already defined at all $k/2^j$, fit a polynomial $\Pi_{j,k}$ to the data $(k'/2^j, F_D(k'/2^j))$ for $k' \in \{(k - (D-1)/2)/2^j, \cdots, (k + (D+1)/2)/2^j\}$– this polynomial is unique–and set

$$F_D((k + 1/2)/2^j) \equiv \Pi_{j,k}((k + 1/2)/2^j).$$

It turns out that this scheme defines a function which is uniformly continuous at the rationals and hence has a unique continuous extension to the reals. This extension defines an (R, D) interpolating wavelet for an $R = R(D)$.

The third family of wavelet is based on de la Vallée Poussin means (Capobianco and Themistoclakis, 2005). The respective kernels have a particularly simple representation and, in the case of Bernstein-Szegö weights, they satisfy a suitable interpolation property. Thus, by restricting to the four types of Chebyshev weights, polynomial interpolating scaling functions are defined by using the de la Vallée Poussin interpolation process, and polynomial wavelets are explicitly given in terms of these scaling functions. Such wavelets are not orthogonal, but they are uniquely determined by the following interpolation constraint: at each resolution level j, both scaling and wavelet functions are interpolating polynomials, and their interpolation knots constitute two disjoint sets whose union gives the interpolation knots of the scaling functions at the higher resolution level $j + 1$.

The above mentioned construction of wavelets are close resemblance to classical wavelet theory. Specifically, the second family closely resemble with the Daubechies wavelet in terms of the construction and in terms of the other concepts, such as smoothness properties etc., but they (interpolating wavelets) are not orthogonal. Hence, the interpolating wavelet F_D is at least as smooth as the corresponding Daubechies wavelet, roughly twice as smooth.

Although, there are some other construction of prewavelets by Buhmann (1994; 1995) and by Blu and Unser (2002) using multiquadric functions based on the classical wavelet theory. Both construction depend on the principle of localization of compactly supported basis functions. Blu and Unser (2002) used fractional splines concept for localization of one–side power function and also multiquadric function while Buhmann (1994) used the 2^{nd} order divided difference equation of multiquadric functions on spaces generated by B–splines. In both cases construction of wavelet basis functions are based on the spaces that are generated by splines, i.e., the wavelet basis functions are basis for the spaces generated by the splines.

1.1.2 Quasi–Interpolating Wavelet Transform

One of the particular interest of progressive reconstruction is multiresolution meshes, where objects are described through hierarchy of increasingly detailed meshes. Each new mesh level is computed from the previous one by first predicting a new point, for instance, by subdivisions schemes, and then correcting the predicted point by a wavelet (or detail) vector. Obviously, this can be generalized to data fitting problem. This problem could be reduced into quasi–interpolating scheme, hence the name quasi–interpolating wavelet, which is described as follows:

Let Ω be a bounded domain in $\mathbb{R}^d$ with minimal smoothness boundary $\partial\Omega$, i.e., we assume that Ω is a Lipschitz domain. By a Lipschitz domain, it means that a bounded and connected open sets whose boundary may be locally described as the graph of a Lipschitz function. The problem of data fitting is formulated in the following way. Given a cloud of data on the boundary, non–coinciding points, denoted by $P = \{(x_i, z_i)\}_{i=1,\cdots,N}$, $x_i \in \Omega$, $z_i \in \mathbb{R}$, for all $i = 1, \cdots, N$, and defined by the corresponding horizontal values $X = \{x_i\}_{i=1,\cdots,N}$, and vertical values $Z = \{z_i\}_{i=1,\cdots,,N}$, one seeks a function $f : \Omega \to \mathbb{R}$ that represents the information contained in P. In this approach, it is to construct a function f of the form

$$f = \sum_{\lambda\in\Lambda} d_\lambda \psi_\lambda,$$

where $\{d_\lambda\}_{\lambda\in\Lambda}$ are vertical off–set vector corresponding to the normal wavelets $\{\psi_\lambda\}_{\lambda\in\Lambda}$ and Λ stands for an appropriate set of indices.

With the above notion, let ϕ be a compactly supported function on Ω such that $\phi : \Omega \to \mathbb{R}$ and $S(\phi)$ be a linear space with generator ϕ, i.e., $S(\phi)$ is the linear space of multi–scaling functions of ϕ. One of the important problem in this area is that construction of quasi–interpolating formulas. It is well known that corresponding to a generator ϕ there are infinitely many quasi–interpolating formulas. A characterization of these formulas, particularly those which depend only on finite discrete set of functions values at different scales are important in the context of approximation. This can be considered as multiresolution. With the aid of this characterization, normal interpolating wavelet transform can be addressed with the following assumptions.

Let (S) be a family of functions in $\mathcal{S}$, for simplicity, assume that the function represented by one–valued functions $y = S(x)$, $-1 \leq x \leq 1$. Similar considerations may apply to unisolvent families for multivariate case. The family (S) is assumed to be n–parametric, solvent, unisolvent, and continuous; more explicitly it is assumed the following:

1. **Solvence:** for any n values $\{x_1, \cdots, x_n\}$ with $-1 \leq x_1 < \cdots < x_n \leq 1$ and arbitrary real numbers $\{y_1, \cdots, y_n\}$ there exists a function S of (S) with $S(x_i) = y_i$, $i = l, \cdots, n$;

2. **Unisolvence:** only one such function exists, in the extended sense that not only, for any two different functions S_0 and S_1 of (S), $S_0 - S_1$ has less than n roots (zeros), but also that this is true if any root x with $|x| < 1$ for which $S_0 - S_1$ does not change sign between $x - \epsilon$ and $x + \epsilon$ is counted as two roots, where $\epsilon > 0$;

3. **Continuity:** $S(x) = S(x; y_1, \cdots, y_n)$ is a continuous function of $x, y_1, \cdots, y_n$.

Remark 1.1.1 *Let $a_1, \cdots, a_{n+1}$ be the vertices of a non–degenerate n simplex of $\mathbb{R}^n$. If $X = \{a_1, \cdots, a_{n+1}\}$ then, as it is well known, X is a 1–unisolvent set for Lagrange interpolation.*

1.2 Problem Statement

Current research concentrates on the applicability of the normal offset concept on real images. In practice, a good initial mesh seems to have crucial impact on the performance. The same idea of normal mesh can be used to select a limited number of crucial coarsest scale samples (pixels). In a normal offset decomposition, the multiscale detail coefficients carry information on the location of the line singularities. The procedure is highly nonlinear. Topological exceptions need to be dealt with carefully. This observation also explains why nonlinear approximation, for compression, is a non-trivial task. Thresholding or tree structured coefficient selection has to deal with the topological aspects.

A first attempt to use normal mesh techniques for image approximation was made by Jansen et al. (2005). Gray–scale images are treated as two dimensional functions dominated by geometric structures comparable with terrain models used in geographical information systems. Authors used normal meshes to approximate piecewise continuous height fields and observe that normal meshes have the capability to adaptively approximate the jump in a way to wedgelets and curvelets. Moreover, authors proved that within certain function class the normal mesh representation achieve an n–term approximation rate using a wavelet transform combined with a nonlinear thresholding $\sigma^2_{L_2}(n) = O(n^{-1})$. Since approximation and compression are tightly related to each other, their results indicate that the normal offset method should be considered for the development of efficient rate–distortion image encoders of piecewise smooth images. Recently normal multiresolution approximation of Geometric Image Approximation was investigated by Aerschot et al. (2009). Further results are available in Aerschot (2009).

In practice of image processing, less smooth spaces, namely BV and the Besov space $B^1_{1,1}$ (Cohen et al., 1999) are used to model gray–scale images. Recall that if Ω is a bounded domain in $\mathbb{R}^d$ then, the Besov space $B^1_{1,1}(\Omega)$ is taken in place of the (large) space $BV(\Omega)$. Both $BV(\Omega)$ and $B^1_{1,1}(\Omega)$ are smoothness spaces of order one in $L_1(\Omega)$, e.g., the space $BV(\Omega)$ is the same as $Lip(1, L_1(\Omega))$. In contrast to BV the $B^1_{1,1}(\Omega)$ norm has a simple equivalent expression as the ℓ_1 norm of the coefficients in a wavelet basis decomposition $f = \sum_{\lambda\in\Lambda} d_\lambda \psi_\lambda$, where Λ denotes the set of indices for the wavelet basis.

1.3 Aims and Objectives

The objective of this research is to investigate the normal multiresolution approximation of a piecewise smooth function by a polynomial, or more generally by a linear combination of basis functions. This research focus on how normal wavelets work for less smooth spaces, particularly spaces that are used to model natural scene images. The optimal algorithms for such applications can be derived from expansions into unconditional bases for the function that models the object to be compressed. To my best of knowledge, so far there has been no representation for normal wavelets that models a object in BV or Besov spaces. Thus, it is essential to study the properties of the normal wavelet coefficients in $B^{\alpha}_{p,q}(\Omega)$, $1 \leq p, q \leq \infty$, for $0 < \alpha < 1$, in order to device a better compression scheme for gray–scale images.

The orthonormal wavelet bases yield unconditional bases for a large variety of function spaces. Probably, the most prominent examples are the (homogeneous) Besov spaces $B^{\alpha}_{p,q}(\mathbb{R})$, $\alpha \in \mathbb{R}$, $0 < p, q < \infty$, which then characterized by decay properties of wavelet coefficients $(\langle f, \psi_{j,k}\rangle)_{j,k\in\mathbb{Z}}$.

To be more precise, define for $\alpha \in \mathbb{R}$, $0 < p, q < \infty$, the coefficient spaces $b^{\alpha}_{p,q}(\mathbb{R})$ as the collection of all complex–valued sequences $t = (t_{j,k})_{j,k\in\mathbb{Z}}$, satisfying

$$\|t\|_{b^{\alpha}_{p,q}(\mathbb{R})} := (\sum_{j\in\mathbb{Z}}(\sum_{k\in\mathbb{Z}}(2^{-j(\alpha+1/2-1/p)}|t_{j,k}|)^p)^{q/p})^{1/q} < \infty. \tag{1.1}$$

It is known that under conditions on the wavelets localization, vanishing moments and smoothness properties, f is in Besov spaces $B^{\alpha}_{p,q}(\mathbb{R})$ if and only if $(\langle f, \psi_{j,k}\rangle)_{j,k\in\mathbb{Z}}$ is in the corresponding coefficient spaces $b^{\alpha}_{p,q}(\mathbb{R})$; see e.g., Frazier et al. (1991),

$$\|\theta\|_{b^{\alpha}_{p,q}(\mathbb{R})} := (\sum_{j\in\mathbb{Z}}(\sum_{k\in\mathbb{Z}}(2^{-j(\alpha+1/2-1/p)}|\theta_{j,k}|)^p)^{q/p})^{1/q} < \infty. \tag{1.2}$$

The above discussion of the continuous setting suggests, using the space of coefficient families $b^{\alpha}_{p,q}(\mathbb{R})$ of non–orthogonal wavelet decomposition, see e.g., Feichtinger and Gröchenig (1992), into normal wavelet decomposition.

1.4 The Scope of this thesis

The scope is two folded :

1. constructing basis functions for Normal Interpolating Wavelet transform and
2. characterizing normal wavelet coefficients in $b^{\alpha}_{p,q}$ for $1 \leq p, q \leq \infty$ and $0 < \alpha < 1$.

The first objective of this research is the construction of quasi–interpolating wavelets on space spanned by univariate multiquadric functions on regularly spaced knot sequences, then generalize to multivariate setting with suitable assumptions. This is similar to linear splines on manifolds. Construction of such basis function is based on principle of interpolation of multiquadric functions. Underlying concept is that normal wavelets are quasi–interpolant which uses the concept as multiquadric at a different resolution level, e.g., multiquadric functions at a different resolution level. A convenient setting of multiresolution analysis is usually based on some sequence of nested closed subspaces of some function spaces, $S(\phi)$ in Banach spaces such that $\mathcal{S}_0 \subset \mathcal{S}_1 \subset \cdots \subset \mathcal{S}_n \subset S(\phi)$, satisfied with the basis function. The best approximation is said to be furnished by that function for which the maximum distance between corresponding points of the approximating and the approximated function is as small as possible. Much of this effort hinges on the idea of multiresolution analysis as a device to construct wavelet approximation likewise traditional wavelet transform.

The method of Radial Basis Functions provides good approximating properties (Wu and Schaback, 1993), a rich theoretical characterization (Buhmann, 2000; Powell, 1992) and a mesh free approach which appears quite natural when working with unorganized data. However, the data reduction strategies (like thinning (Floater and Iske, 1996a) or adaptive thinning (Dyn et al., 2002)) that are necessary to construct a multiscale formulation of the problem, as in (Floater and Iske, 1996b), or simply to make it tractable usually enforce the construction of some auxiliary triangulation. The error of a normal mesh approximation in 2 D is completely dominated by the error of this piecewise linear approximation of the geometry information in the edge. This observation suggests that curved triangles have the potential of catching the geometry information even better. It is shown that the objective is directly related with auxiliary triangulation with multiquadric functions.

With regard to the second objective, it is shown that the above decomposition is unconditional basis for the function space that models the object. Which spaces of functions other than $L_p(\mathbb{R})$ should we consider? It is supposed to deduce smoothness properties of a function from its normal wavelet coefficients. This is similar to measure smoothness properties of a function from its samples. Thus, it is meaningful to consider smoothness properties such as size, growth and oscillation. We are then led to look at Besov–type spaces of functions in $\Omega \subset \mathbb{R}^d$.

1.5 Contribution

In Chapter 3 it is shown that the Normal Multiresolution Triangulation (NMT) Interpolation could be interpolated with multiquadric functions under some certain restrictions. Moreover, it is shown in Chapter 7 that this interpolation scheme could be used in image compression scheme effectively. In practice, digital images are samples on square grids, hence that, using normal multiresolution approximation using triangular meshes requires remeshing operation. Then there should be a remeshing operation to display the image encoded with normal mesh. Such remeshing is not needed in Normal Multiresolution Interpolation to display the image.

In Chapter 4 non–orthogonal wavelet expansions associated with a class of mother wavelets on real line $\mathbb{R}$ is considered. This class of wavelets comprises mother wavelets that are not necessarily integrable over the whole real line. The coefficients are determined by quasi–interpolation formula rather than sampling. The pointwise convergence of these wavelet expansions (Zayed, 2000) is investigated.

In Chapter 5 the above non–orthogonal wavelet is constructed in a bounded domain $\Omega \subset \mathbb{R}^d$, $d \geq 1$, with uniform cone condition. An important achievement is that curved triangles are used as basis function in higher dimension. Moreover, L_p stability of the above decomposition is proved with the concept of admissible triangulation sequence property. This scheme does not require tenser product of wavelets on real line. This is an natural extension of wavelet on real line.

Remarkably, admissible function sequence is introduced in the Chapter 5. This leads to characterization of smooth function on a non–smooth discrete objects. This has been developed in Chapter 6 by atomic decomposition of functions spaces (Frazier and Jawerth, 1985, 1990; Frazier et al., 1991) with normal interpolating wavelets. The nonlinear character of normal multiresolution approximation itself makes it harder to analyze the effect of nonlinear approximation of images, but with the decomposition into function spaces it is possible analyze the effect of nonlinear approximation. Moreover, nonlinear approximation (DeVore, 1998; DeVore and Popov, 1988) makes it possible to thresholding or tree structured coefficients selection. Topological considerations is not necessary as in the NMT approximation (Lounsbery et al., 1997).

1.6 Structure of this Dissertation

The organization of this dissertation is as follows. There now follows a chapter with preliminaries, where normal multiresolution approximation algorithm and essentially some preliminaries needed in later chapters are described. In Chapter 3, a basis function using Hardy's multiquadric function is introduced for normal multiresolution approximation. Chapter 4 present the normal interpolating wavelet transform on real line using quasi–interpolating wavelet. Based on the above univariate interpolating basis function, the transformation is generalized to multivariate setting in Chapter 5 and, Chapter 6 is devoted to the decomposition of function into normal wavelet with the assumed smooth properties in to Besov spaces. In Chapter 7, an specific application such as image compression is discussed. Finally Chapter 8 is devoted to discussions on diverse issues concerning with the normal wavelet interpolation.

Chapter 2

Background

2.1 Multiresolution

The idea of a multiresolution analysis is a hierarchy of averages and the study of their difference, devised by the algorithm of image analysis and reconstruction, by Laplacian Pyramid scheme of Burt and Adelson, (1983). The Lagrangian interpolation function is the basis of Mallat (1989) ideas to view orthonormal wavelet bases as the tool for multiresolution analysis. Following this development, several authors carried out detailed analysis in construction of wavelets and the properties of multiresolution analysis. Essentially wavelets are building blocks to represent data and functions in different resolutions. In this context, the recent development of wavelet theory has been given giant leap towards local scale decompositions.

A multiresolution approximation is the decomposition of a function, e.g., $f(x)$, into scales. A discrete multiresolution analysis is a decomposition of a vector array $(f_j)_{j\in\mathbb{Z}}$ representing point value discretization of $f(x)$ on increasing sequence of partitions say $\{x_j\}_{j\in\mathbb{Z}}$ on a set $\partial\Omega$, where $\partial\Omega$ is the boundary of a bounded domain $\Omega \subset \mathbb{R}^d$. A discrete multiresolution algorithm consists of a decomposition (analysis) which generates the scale coefficients from the input array and reconstruction (synthesis) which recovers the input array from the scale coefficients.

2.1.1 Wavelet

A convenient setting of multiresolution analysis is the wavelet. Traditionally wavelets are square integrable functions $\psi_{j,k}$ defined as translates and dilates of one particular function, the mother wavelet ψ. These functions form orthonormal basis of some subspaces V_j, $j \in \mathbb{Z}$, of $L_2(\mathbb{R})$.

Wavelets are versatile tool for representing general functions and data sets, and they enjoy widespread use in areas as diverse as signal processing, image compression, speech synthesis, finite element methods, and statistical analysis (among many others). The particular appeal of wavelets derives from their representational and computational efficiency: most data sets exhibit correlation both in space and frequency as well as other type of structures, such as geometric properties of an object. These can be modeled with high accuracy through sparse representation of wavelet coefficients. Wavelet representation can also computed fast because they are built on multiresolution property. The mathematical properties of wavelets are well understood in view of functional setting, i.e., for the approximation of function of one or more variables.

Surfaces in Computer–Aided Geometric Design representation of 3 D objects, e.g., human face recognition, are modeled with millions of control points. These points can for instance arise from Laser Scanners. Hence, the representation of 3 D geometric objects that allow for efficient computational processing have become an increasingly important problem. It is possible to sample real world 3 D objects with a very high level of details; scanners can generate huge amount of data typically in the form of triangular meshes with complex topology. The irregular format makes processing like compression, denoising, filtering and texturing, difficult. New ways of describing 3 D objects can lead to significantly improved compressing algorithms. In addition, it is often desirable to support progressive reconstruction: a coarse version of the object is first quickly reconstructed and additional levels of detail are added as the reconstruction continues. This is useful in streaming application in networked environments. Only the wavelet vectors needed are stored depending on the smoothness of the surface or curve and because most of the wavelet vectors are small, which leads to compression.

Reference to Daubechies et al. (1999), for progressive reconstruction, multiscale reconstruction is of particular interest, where the objects were described through an hierarchy of increasing wavelet coefficients. Wavelets at level j are typically used, in the regular case, as the building blocks to represent any function in the multiresolution analysis that lies in the $j+1$–th approximation space V_{j+1}, but not in the coarser approximation space $V_j \subset V_{j+1}$. One can introduce the same concept in irregular setting as well. The scaling functions $\varphi_{j,k}$'s are limit functions obtained from starting the subdivision at level j, from the initial data $f_{j,l} = \delta_{l,k}$, and refining from there on. Under appropriate assumption on the subdivision operators S_j, the $\varphi_{j,k}$'s are independent; V_j is the function space spanned by them. Clearly $V_j \subset V_{j+1}$. As in the irregular case, there are many different reasonable choices for complement spaces W_j (which will be spanned by the wavelets at level j) that satisfy $V_{j+1} = V_j \oplus W_j$. When the scaling functions are interpolating as in Lagrangian case, i.e., $\varphi_{j,k}(x_{j,k'}) = \delta_{k,k'}$, then the simple choice for wavelet is given by $\psi_{j,m} = \varphi_{j+1,2m+1}$, i.e., the wavelet is simply a finer scale scaling function at an odd location. This is called as an interpolating wavelet. This has been modified in the following manner.

Hypothesis: This structure is quite similar to a wavelet multiresolution analysis with $S(\phi)$ playing the role of the set of wavelet scaling functions. It lacks the explicit orthogonal (or bi–orthogonal) structure of wavelets but has much more flexibility. The main idea behind this concept is: **Locally supported wavelets are obtained by relaxing the condition that the wavelets should lie in the orthogonal complement spaces.** This is immediately related to the techniques such as lifting scheme (Sweldens, 1997) and normal multiresolution wavelets (Daubechies et al., 2004). Moreover, the wavelet transform is non–orthogonal projection onto the nested sub–spaces and satisfies the property $Qp = p$, where p is the polynomial of order less than or equal to d and d is odd.

Each level of reconstruction is computed with scaling functions and wavelet coefficients. The scaling functions are triangular functions represent normal multiresolution wavelets. Normal multiresolution wavelets are quasi–interpolating basis functions. The basis function is a piecewise liner function which satisfying Lagrangian interpolating condition, and it would be shown as a multiquadric function. Clearly, multiquadric function is a shift–invariant. The quasi–interpolating basis function ϕ is constructed by some suitable linear combination of multiquadric function φ, which spans the subspaces V_j, $j = 0, 1, \cdots$, of some Banach spaces $\mathcal{B}$. These subspaces satisfying the nested property such that $V_j \subset V_{j+1}$. Hence, by way of construction the quasi–interpolating function which span the same spaces V_j, i.e., generating function. By the above concept $\theta_{j,k} = \phi_{j+1,2k+1}$, i.e., the wavelet is simply a finer scale scaling function at an odd location $\phi_{j,k}(x_{j,k'}) = \delta_{k,k'}$. Thus, we have a quasi–interpolating wavelet.

2.2 Related Work

Many first generation wavelet families have been constructed over the last ten years. Except for Donoho (1992), they all rely on the Fourier transform as a basic construction tool. The reason is that translation and dilation become algebraic operations in the Fourier domain. In fact, in the early 1980s, several years before the above developments, Strömberg (1981) discovered the first orthogonal wavelets, with a technique based on spline interpolation which does not rely on the Fourier transform.

Wavelets form a versatile tool for representing general functions or data sets. Essentially we can think of them as data building blocks. Their fundamental property is that they allow for representations which are efficient and which can be computed fast. Quoted from Donoho (1993a), wavelets are optimal bases for compressing, estimating, and recovering functions in different function spaces $\mathcal{F}$. Roughly speaking, for a general class of functions, the essential information contained in a function is captured by a small fraction of the wavelet coefficients. Wavelets are based in sub–division scheme and traditionally wavelets are functions of $\phi_{i,j}$ defined as the translates and dilates of one or more particular function, the mother wavelet ϕ. This is known as first generation wavelets.

As it is mentioned earlier the sub–division scheme determine the wavelets coefficients and these wavelets are represented in regular spaced points. Hence, it does not reap good results in general settings such as with real data. Moreover, wavelets need not to be translates and dilates of one or more templates. Generalization of this type is called second generation wavelets (Sweldens, 1997). In this research it is concerned with a more general setting but, still enjoy all the powerful properties of first generation wavelets. Several results concerning to the construction of wavelets are adapted to some of these cases in second generation already exist. For example, wavelets on an interval (Chui and Quak, 1992; Cohen et al., 1993a,b), wavelets on bounded domains (Cohen et al., 2000), spline wavelets for irregular samples (Dahmen and Micchelli, 1993) and weighted wavelets. These constructions are tailored toward one specific setting. Other instances of second generation wavelets have been reported in the literatures, e.g., the construction of scaling functions through subdivision (Dahmen, 1991), basis constructions (Dahmen et al., 1994a), as well as the development of stability criteria (Dahmen, 1991, 1994b).

Normal Multiresolution Approximation: Multiresolution approximation of a curve or surface is normal if all the wavelet vectors perfectly align with a locally defined normal direction which depends on the coarser levels. Given that this normal direction only depends on coarser levels, only a single scalar coefficient needs to be stored instead of the standard 2–or 3 vectors. Clearly this is extremely useful for compression applications (Khodakovsky et al., 2000). An algorithm in Guskov et al. (2000) gives to build normal mesh approximation of large complex scanned geometry. All these methods really refer to isoscale triangulation of a function in a bounded domain $\Omega \subset \mathbb{R}^d$, $d \geq 1$, and data sets are on its minimal smooth boundary $\partial\Omega$.

Triangulation: Triangulations consist of triangles, that is, triple of vertices connected by edges of the triangle. These triangles have potential to represent arbitrary edges and contours. More accurately, with a fewer number of patches than a fixed square representation. For efficient processing of 3 D mesh data, multiscale triangulations based on nonlinear subdivision has been proposed in computer graphics. Multiresolution triangulations meshes are widely used in computer graphics for representing 3 D shapes and 2 D piecewise smooth functions such as gray scale images, because triangles have potential to be more efficient approximation at the discontinuities between the smooth pieces than the other standard tools like wavelets.

In multiresolution meshes, objects are described through hierarchy of increasingly detailed meshes. Each new mesh level is computed from the previous one by first predicting a new point. The prediction is done by a subdivision scheme, such as Butterfly (Loop, 1987, 1994), the predicted point would be corrected by wavelet (detail) vector, i.e., these wavelets need to be stored. In general setting one should be able to characterize these wavelets (second generation) in various functional spaces of interest. The following section presents an overview of subdivision scheme based on Daubechies et al. (2004) and Runborg (2005).

2.3 Normal Multiresolution Approximation

Guskov et al. (2000) describe the idea of normal multiresolution approximation based on the midpoint subdivision scheme. Subdivision schemes produce millions of control parameters for representing such curves and surfaces. These control parameters can be reduced by combining with wavelets. For an optimal representation of curves and surfaces, normal mesh subdivision scheme is represented as a multiresolution triangulation over the different level of resolutions. The normal difference between the two subsequent level of resolutions is called as normal wavelet coefficient. The original curve Γ is described by successively finer approximations which are organized in different multiresolution layers indexed by j, see Figure 2.1. It is assumed that the Γ is a continuous and non intersecting curve in the plane, whose endpoints are to be the 0^{th} level multiresolution points $v_{0,0}$ and $v_{0,1}$. To construct the vertices at level $(j+1)$, we first set $v_{j+1,2k} = v_{j,k}$; this is what makes the construction interpolating. Also compute new points $v_{j+1,2k+1}$; each $v_{j+1,2k+1}$ lies in between the two old points $v_{j,k}$ and $v_{j,k+1}$. This is done by first computing a predicted or

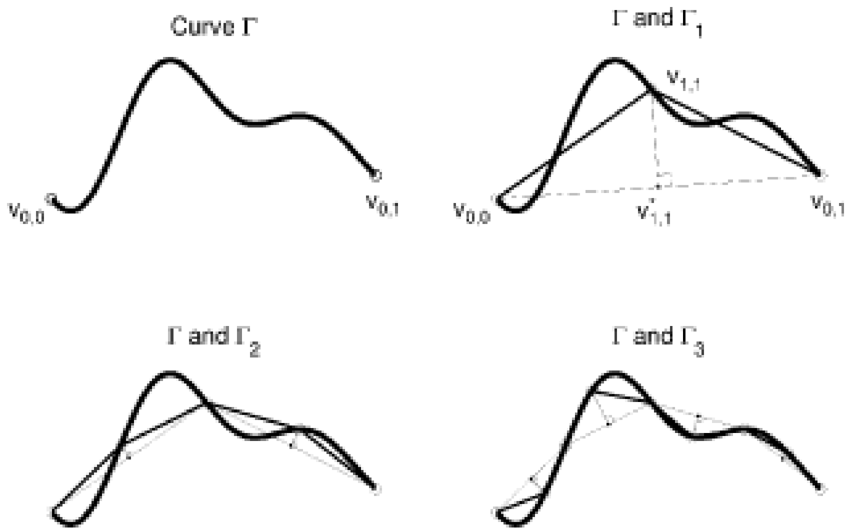


Figure 2.1: Normal mesh algorithm using the mean value of adjacent points as predictor.

base point as the mean value of the old points, $v^*_{j+1,2k+1} = (v_{j,k} + v_{j,k+1})/2$. Next draw a line from $v^*_{j+1,2k+1}$ in the direction orthogonal to the line segment $(v_{j,k}, v_{j,k+1})$. This line is guaranteed to cross the curve segment between $v_{j,k}$ and $v_{j,k+1}$ at least once and we call one of these points $v_{j+1,2k+1}$.

As this procedure continues, the normal polyline Γ_j , i.e. the piecewise linear curve connecting the points $v_{j,k}$ comes closer and closer to Γ. Now think of this as a wavelet transformation similar to the notion of lifting (Sweldens, 1997). Think of $v^*_{j+1,2k+1}$ as a prediction of the real point $v_{j+1,2k+1}$ computed based only on coarser information. Then, $w_{j,k} = \|v_{j+1,2k+1} - v^*_{j+1,2k+1}\|$ is a wavelet vector. Given that this vector points in a direction normal to a segment that again only depends on coarser data. We only need to store the length and one sign bit for this normal component to characterize it completely. Thus, we have a polyline with no parameter information. One can also consider normal polylines with respect to fancier predictors. For example, one could compute a base point and normal estimate using the well known 4 point rule. Essentially, any predictor which only depends on the coarser level is allowed. Hence, a normal polyline is completely determined by a scalar component per vertex.

Normal polylines are closely related to certain well known fractal curves such as the Koch Snowflake, see Figure 2.2. Here each time a line segment is divided into three sub-segments. The left and right get a normal coefficient of zero while the middle receives a normal coefficient, as such the resulting triangle is isoscale. Hence, the polylines leading to snowflake with respect to the midpoint subdivision. There is also a close connection with wavelets. The normal wavelet coefficients can be seen as a piecewise linear wavelet

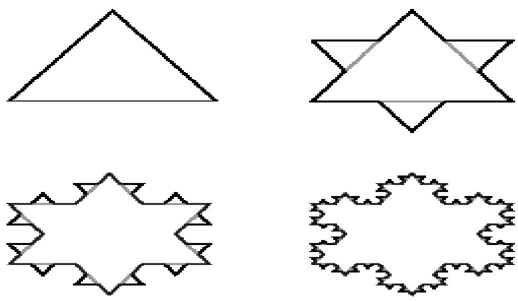


Figure 2.2: Koch Snowflake

transform of the original curve. Because, the tangential components are always zero, there are half as many wavelet coefficients as the original scalar coefficients. Thus, one saves 50% memory right away. In addition to that, the wavelets have their usual decorrelation properties. In the functional case, the above transform corresponds to an uplifted interpolating piecewise linear wavelet transform as introduced by Donoho (1992). There, it is shown that interpolating wavelets with no primal, but many dual moments are well suited for smooth functions. Unlike in the function setting, not all wavelets from the same level j have the same physical scale. Here the scale of each coefficient is essentially the length of the base of its triangle.

The above method can be generalized with more general methods, such as Lagrangian subdivision scheme, which will lead to higher quality approximation for smooth curves. An important observation is that the above scheme interpolates a continuous function $f : \mathbb{R} \to \mathbb{C}$ at a set of control points $X_i = \{x_{i_1}, x_{i_2}, \cdots, x_{i_n}\}$ in $\Omega \subseteq \mathbb{R}$ at different resolution levels.

2.4 Generalized Subdivision Scheme

Generalization of wavelet construction to the non–traditional settings, such as lifting and normal multiresolution scheme, used the generalized subdivision schemes. Subdivision schemes provide fast algorithms, create natural multiresolution structures and yield underlying scaling functions and wavelets. Subdivision techniques are generally used to build smooth functions starting from coarse description to finer. The main idea behind the subdivision scheme is the iteration of up–sampling and local averaging to build functions. Subdivisions schemes are studied in computer aided geometric design to intricate geometrical shapes, in the context of corner cutting, construction of piecewise polynomial curves and algorithms of iterative spline generators. Later splines were studied in spline functions and wavelets.

2.4.1 Sequences

Let X denote the space of infinite sequences. Sequences will be written in bold face, and elements of sequences in normal font, $\mathbf{x} := (x_k)_k$ or simply $(x_k)_k$. Define the difference operator Δ as

$$(\Delta x)_k = x_{k+1} - x_k. \tag{2.1}$$

Often a sequence itself is indexed by the subdivision level j; then use the convention that $\mathbf{x} := (x_{j,k})$. Think of a sequence at level j as associated with the parameters $t_{j,k} = k2^{-j}$. Therefore, also define the divided difference operator $D_j = 2^j \Delta$. The divided differences of a sequence x_j are

$$x_j^{[p]} = D_j^p x, \quad p > 0.$$

Use the usual sup–norm on X, $|x|_\infty = \sup_k |x_k|$. Scalar functions can be applied to sequences component wise, so that $(F(x))_k = F(x_k)$. Use the special sequence $\mathbf{k} = (k)$, i.e., the sequence of the k–th entry is k itself. The sequence with all entries equal to 0 is $\mathbf{0}$, similarly the sequence with all entries equal to 1 is $\mathbf{1}$.

2.4.2 Local Stationary Subdivision

A local stationary subdivision scheme is characterized by a bounded linear operator S from X to itself, defined by a finite sequence s as follows:

$$(Sx)_k = \sum_l s_{k-2l} x_l. \tag{2.2}$$

The width B of S is defined by $B = [[(k - B)/2, [(k + B)/2]]$. Given S, we can apply it iteratively starting from a sequence a_0 and define for all $j \geq 0$,

$$a_{j+1} = Sa_j.$$

The sequence a_0 can be viewed as a coarse approximation of a function on the integer grid; the sequence a_j then gives successively finer approximation of the function on grids with spacing 2^{-j}. A subdivision is interpolating if $s_{2l} = \delta_{l,0}$, implying $a_{j+1,2k} = a_{j,k}$ for all j, k; in this case $a_{j,k}$ interpreted as function values of f, $a_{j,k} = f(t_{j,k}) = f(2^{-j}k)$.

The order of a subdivision scheme S is the largest degree for which it leaves the corresponding space of monotonic polynomial invariant. More precisely, S is of order P, if P is largest integer such that for all p–degree monotonic polynomials P with $0 \leq p \leq P$, a p–degree monotonic polynomial Q exists so that $SP(k) = Q(k/2)$. If S is interpolating, then $SP(k) = P(k/2)$. Always assume that P is at least one so that, $S1 = 1$. The derived subdivision schemes are defined as

$$S^0 = S, \quad S^{[p]} = 2\Delta S^{[p-1]}\Delta^{-1}, \quad p > 0.$$

Note that $S^{[p]}$ is well defined as long as $S^{[p-1]}$ has at least order one, and that the order $S^{[p]}$ is one less than the order of $S^{[p-1]}$. Thus $S^{[p]}$ is defined for $p \leq P$. Also note that

$$S^{[p]}D_j = D_{j+1}S^{[p-1]} \quad and \quad S^{[p]}D_j^p = D_{j+1}^p S.$$

The special example is the midpoint interpolating subdivision scheme $S = S_2$. This scheme has the order $P = 2$ and yields piecewise linear limit functions.

2.4.3 Technical Preliminaries

Let $C^0(I)$ be the continuous and bounded functions defined on a (possibly unbounded) interval $I \subseteq \mathbb{R}$. Moreover, for a positive integer p, let $C^p(I)$ be constituted by the functions in $C^0(I)$ with a p^{th}–derivative that is continuous and bounded on I. The notation for fractional regularity is as follows. For $f \in C^0(I)$, let

$$\Omega(r, f) = \sup_{t_0, t_1 \in I} \frac{|f(t_0) - f(t_1)|}{|t_0 - t_1|^r}.$$

For $p \in \mathbb{N}$ and $0 < r < 1$ define the class $C^{p+r}(I)$ as the set of functions $f \in C^p(I)$ for which $\Omega(r, f^p)$ is bounded. Similarly, use the notation $f \in Lip^\alpha(I)$, with $\alpha = p + r$, $p \in \mathbb{R}$, $0 < r \leq 1$, when $f \in C^p(I)$ and $\Omega(r, f^p)$ is bounded. For $\alpha \notin \mathbb{N}$, the spaces $Lip^\alpha(I)$ and $C^0(I)$ coincide; for $\alpha \in \mathbb{N}$, however, $C^\alpha(I) \nsubseteq Lip^\alpha(I)$. Finally, $C^{\alpha^-}(I)$ or $Lip^{\alpha^-}(I)$ stands for

$$\bigcap_{\alpha' < \alpha} C^{\alpha'}(I) = \bigcap_{\alpha' < \alpha} Lip^{\alpha'}(I).$$

We shall use the notation α^- in more general contexts as well. More precisely, if r is a real number, we shall use the notation r' wherever we could insert in its place $r - \epsilon$ with

$\epsilon > 0$ arbitrarily small. With some abuse of notation we adopt the conventions $r^- < r'$ if $r < r'$ and $r < r'^-$ if $r < r'$. It follows that $min(r^-, r')$ equals r' if $r > r'$ and r^- if $r < r'$.

Taylor's theorem says that if $f \in Lip^\alpha(I)$ and $p \in \mathbb{N}$, $0 < r \leq 1$ we can write

$$f(x) = \sum_{k=0}^{p} \frac{f^k(x_0)}{k!}(x - x_0)^k + R(x),$$

where the remainder term $R(x)$ is bounded by

$$|R(x)| \leq \frac{\Omega(r, f^p)}{p!}|x - x_0|^{p+r}, \quad \forall x, x_0 \in I.$$

The following theorem concerns (Daubechies et al., 2004) the sequences x_j that are not formed exactly by subdivision, but that are close in the sense that the difference between x_j and Sx_{j-1} goes to zero exponentially.

Theorem 2.4.1 *Let S be a subdivision scheme of order $P \geq 1$ and $S^{[p]}$ its p^{th} derived scheme, with $p \leq P$. Assume there are positive real numbers C, μ such that*

$$|S^{[p]^j}|_\infty \leq C2^{\mu j}, \quad \forall j \geq 0.$$

Let $\{x_j\}$ be a family of sequences satisfying

$$|x_{j+1} - Sx_j|_\infty \leq C2^{-\nu j}, \quad j \geq 0,$$

for some real number ν and let $\varphi_j(t)$ be a piecewise linear function interpolating the points $(x_{j,k})$ at $t = k2^{-j}$ for all j, k. Set

$$P + \kappa := min(p - \mu, \nu), \quad P \in \mathbb{N}, \quad 0 < \kappa \leq 1.$$

If $P \geq 0$ and $|x_0|_\infty < \infty$, then there exists a function $\varphi \in C^{(P+\kappa)-}(\mathbb{R})$ such that $\varphi_j(t) \to \varphi$ uniformly exponentially.

This theorem says that the regularity of the limit function of a family of sequences approximately generated by subdivision is bounded both by the regularity of the subdivision scheme and the speed of the approximation.

Note that this is similar to standard results linking smoothness of functions with the decay of their wavelet coefficients, where the wavelet coefficients at level j correspond to the difference between x_{j+1} and Sx_j; in the wavelet case, the subdivision operator S is determined by the low–pass filter corresponding to the wavelet basis. Next we go into more detail on the construction of a normal multiresolution for a smooth curve Γ in the plane. Even though the normal multiresolution algorithm does not depend on any parametrization, to formulate the theorem it is convenient to parametrize Γ by one of the x– or y–coordinates. A piecewise C^1 curve can always be broken up into adjacent finite length pieces, possibly overlapping, that can be well parametrized by the x–coordinate or by the y–coordinate; by restricting ourselves to these different pieces separately, and interchanging the names of the two coordinates, we may thus assume that the curve Γ is parametrized by its x–coordinate, so that

$$\Gamma = \{(x, \gamma(x)); x \in I\},$$

where γ is a smooth function and I is a bounded interval. The above theorems are then applied to each of these pieces individually and yielding results valid for the whole curve. There is a complication when the number of pieces is infinite, since the assumptions in the theorem might not hold uniformly, and this case is not considered. In general, assume that γ is at least $C^1(I)$, but occasionally consider the more general case where γ is Hölder

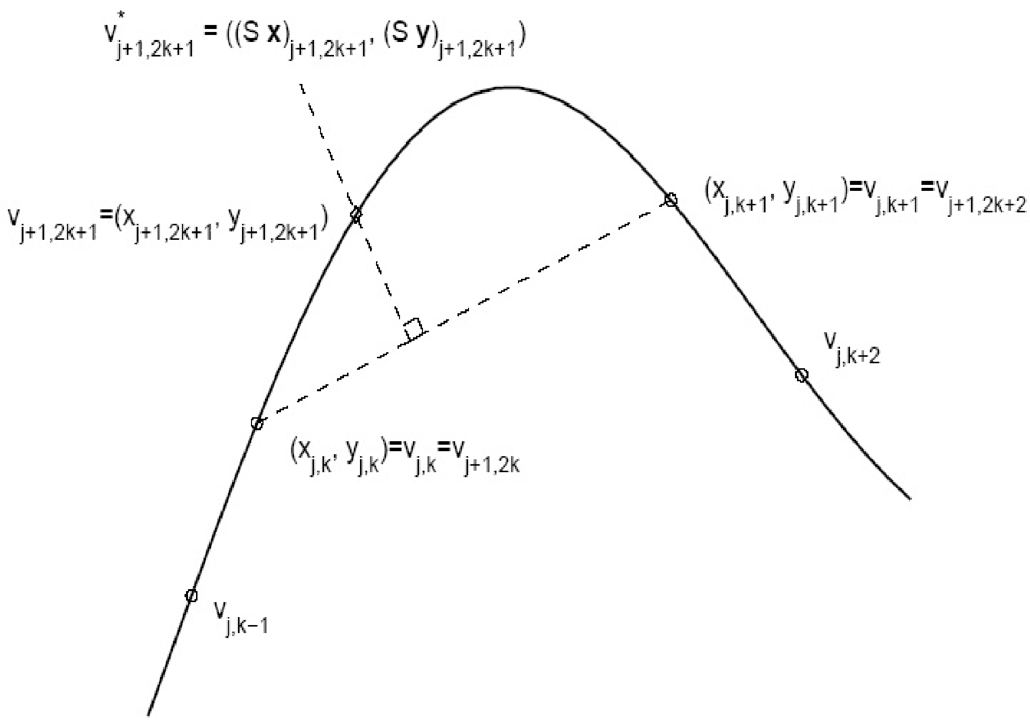


Figure 2.3: Notation for the normal scheme.

continuous with exponent β, where $0 < \beta \leq 1$.

Having reduced Γ (at least locally) to the graph of a function $\gamma(x)$, we can rephrase the basic step in the construction of a normal multiresolution given in Figure 2.3. Start with a sequence x_j on level j and define $y_j = \gamma(x_j)$. Next, use an interpolating subdivision scheme S to compute the sequences $x^*_{j+1} = Sx_j$ and $y^*_{j+1} = Sy_j$. In general, y^*_{j+1} is not equal to $\gamma(x^*_{j+1})$, but we will see they are close. Next draw the line through $(x^*_{j+1,2k+1}, y^*_{j+1,2k+1})$ that is perpendicular to the line connecting $(x_{j,k}, y_{j,k})$ and $(x_{j,k+1}, y_{j,k+1})$. This line and the piece of Γ, between $(x_{j,k}, y_{j,k})$ and $(x_{j,k+1}, y_{j,k+1})$ have to intersect at least at one point. We choose one of the intersection points to be the new point $(x_{j+1,2k+1}, y_{j+1,2k+1} = \gamma(x_{j+1,2k+1}))$ (assume that there is a rule established which uniquely picks out one of the solutions; pick the solution closest to the predicted point). Given that y_j is always $\gamma(x_j)$, we focus our attention on the convergence of the x_j sequences. We will call a family of sequences $\{x_j\}$ defined by the above procedure, a family of sequences generated by the (S, γ) normal scheme.

To have a proper parametrization, we need that all x_j sequences are increasing, i.e.,$\Delta x_j > 0$. In general, there are very few subdivision schemes that always preserve increasing sequences. In this case, the x_j are obtained by a nonlinear perturbation of subdivision so the situation is even more complex. Fortunately, there are conditions on both the subdivision scheme and the initial sequence that guarantee that the x_j will be increasing. The following theorem introduces a non uniformity measure N of a sequence which is the maximal ratio of the length of two neighboring intervals; it states that if the non uniformity of the initial sequence is bounded and the subdivision scheme preserves this bound, the sequences x_j generated by the normal scheme will be increasing and converge exponentially.

Theorem 2.4.2 *Let S be an interpolating subdivision scheme. Let the non uniformity $N(x)$ be defined by*

$$N(x) := \sup_k \max\left(\frac{|(\Delta x)_k|}{|(\Delta x)_{k+1}|}, \frac{|(\Delta x)_{k+1}|}{|(\Delta x)_k|}\right). \tag{2.3}$$

Suppose there is an ρ such that for every strictly increasing x with $N(x) \leq \rho$, Sx is strictly increasing as well, and satisfies $N(Sx) \leq N(x)$. Suppose x_0 is strictly increasing, with sufficiently small $|\Delta x_0|_\infty$ and $N(x_0) < \rho$. If $\gamma \in C^2(\mathbb{R})$, then x_j is strictly increasing for *all j , with $N(x_j) \leq \rho$ for all j , and the x_j converge exponentially, i.e., there is a $\delta < 1$, so that,*

$$|\Delta x_j|_\infty \leq \delta^j |\Delta x_0|_\infty, \quad for\ all\ j \geq 0.$$

If S is the midpoint interpolating scheme, for which $(Sxj)_{2k+1} = (x_{j,k} + x_{j,k+1})/2$, the same conclusions follow if γ is merely Lipschitz continuous, without the smallness assumptions on $|\Delta x_0|_\infty$ and $N(x_0)$.

Examples of subdivision schemes that meet the requirements in the theorem are, for instance, the first Lagrange interpolation schemes.

One of the important features of a normal multiresolution is the decay of the offsets in each of the normal directions. We will refer to these as wavelet coefficients $\omega_{j,k}$ which are defined as

$$\omega_{j,k} = \sqrt{(x_{j+1,2k+1} - x^*_{j+1,2k+1})^2 + (y_{j+1,2k+1} - y^*_{j+1,2k+1})^2}.$$

The rate of convergence to 0 of the wavelet coefficients is then determined by the order of P and the regularity of S, and the smoothness of Γ.

Theorem 2.4.3 *Let S be an interpolating subdivision scheme of order $P \geq 1$ and $S^{[p]}$ its p^{th}–derived scheme, with $p \leq P$. Assume there are positive real numbers C, μ such that*

$$|S^{[p]^j}|_\infty \leq C 2^{\mu j}, \quad \forall j \geq 0, \quad \mu \leq p-1.$$

Let $\{x_j\}$ be a family of increasing sequences generated by the (S, γ) normal scheme for which there is a $\delta < 1$ such that

$$|\Delta x_j|_\infty \leq C\delta^j.$$

Let $x_j(t)$ be a piecewise linear function interpolating the points $x_{j,k}$ at $t = k2^{-j} \in [0,1]$. If $\gamma \in C^\beta(\mathbb{R})$ with $\beta \geq 2$ then $x_j(t)$ converges uniformly exponentially to $x(t)$ and $x \in C^{Q-}([0,1])$, where $Q := min(p-\mu, \beta)$. In addition, let $Q' := \min(p-\mu+1, \beta, P)$. Then, for all $\epsilon > 0$ there is a constant C_ϵ for which the wavelet coefficients,

$$\omega_{j,k} = \sqrt{(x_{j+1,2k+1} - (Sx)^*_{2k+1})^2 + (y_{j+1,2k+1} - (S\gamma(x_j))_{2k+1})^2},$$

satisfy

$$|w_j|_\infty \leq C_\epsilon 2^{-j(Q'-\epsilon)}.$$

Finally, if $Q > 1$, let $Q'' = min(Q - 1, 1)$. Then for sufficiently large j and arbitrary $\epsilon > 0$, there is a constant C_ϵ such that

$$N(x_j) - 1 \leq C_\epsilon 2^{-j(Q' - \epsilon)},$$

with $N(x_j)$ defined as in (2.3).

Above theorems are obtained from Daubechies et al. (2004) in order to derive the following Remark 2.4.1.

Remark 2.4.1 *Suppose $S = S_2$, $\beta > 0$ and x_0 is strictly increasing. Then, x_j is strictly increasing for all $j \geq 0$. If $\gamma \in C^\beta(\mathbb{R})$ with $0 < \beta < 1$ then there is a C such that*

$$\|\Delta x_j\|_\infty \leq \frac{C}{1 + j^{\frac{\beta}{1-\beta}}}, \quad \forall j \geq 0. \tag{2.4}$$

If $\gamma \in Lip^1(\mathbb{R})$ then there is a $\delta < 1$ such that

$$\|\Delta x_j\|_\infty \leq \delta^j |\Delta x_0|_\infty, \quad \forall j \geq 0. \tag{2.5}$$

In both cases there is a constant c such that

$$\|\omega_{j,k}\|_\infty \leq c 2^{-j\beta}, \tag{2.6}$$

for sufficiently large $j \geq j_0$, i.e., wavelet coefficients are depend on the regularity of the interpolating function $\gamma_j(t) \to \gamma(t)$ as $j \to \infty$. ***Thus, we have the sequence of interpolating functions $\gamma_j(t)$ as $j \to \infty$.***

2.5 Summary

Finally, before concluding this chapter, remark that there is a interpolating function for normal subdivision scheme which uniformly converges to the function f being interpolated. The order of convergence is depend on the smoothness of the function f which is $O(2^{-j\beta})$, where $0 < \beta \leq 1$ is the smoothness of the function f, i.e., f satisfy the property of Lipschitz condition. Throughout this dissertation it is assumed that the the function f satisfy the property of Lipschitz condition except it is explicitly stated therein. It is clear from the above the function be approximated by differencing with fractal nature.

It is shown that the Lagrange interpolation schemes meets the normal subdivision scheme. In the next chapter, an interpolating basis function based on the Lagrangian is developed. The basis function is Hardy's multiquadric function since it admits the Lagrangian interpolation polynomials.

Chapter 3

Normal Multiresolution Interpolation

3.1 Interpolation Methods

A standard problem in many applications requires one to find a reconstruction of a function f from a collection of samples $f(x_n)$. In most applications the assumption of f is band–limited, or equivalently that f is an entire function of exponential type, is well justified, and frequently the sampling points are non–uniformly spaced or distributed quite randomly. Then the mathematical problem is to find conditions under which f can be reconstructed completely from its samples $f(x_n)$.

In particular, from the previous chapter we know that normal multiresolution approximation is an quasi–interpolating approximation on multiresolution. Thus, the normal multiresolution approximation has Interpolating Basis Functions for each and every resolution. Hence, our immediate objective is to find interpolating basis functions for each resolution level in a principal shift–invariant spaces which are nested. This has been generalized as multivariate data interpolation problems; we are usually given data (x_j, f_j), $j = 1, \cdots, N$ with distinct $x_j \in \mathbb{R}^s$ and $f_j \in \mathbb{R}$, and we want to find a (continuous) function $P_f : \mathbb{R}^s \to \mathbb{R}$ such that

$$P_f(x_j) = f_j, \quad j = 1, \cdots, N. \tag{3.1}$$

For classical interpolation methods we assume P_f to be a linear combination of a set of basis functions ϕ_j, i.e.,

$$P_f(x) = \sum_{j=1}^{N} c_j \phi_j(x). \tag{3.2}$$

The coefficients c_j are determined by satisfying the constraint (3.1). To guarantee the existence of a unique set of c_j for arbitrary distinct x_j, it is known that, when $s > 1$ and $N > 1$, ϕ_j must be x_j–dependent. In literatures, the ϕ 's are frequently chosen to be basis functions generated by shifts of a single strictly positive definite basic function ϕ, i.e.,

$$\phi_j(x) = \phi(x - x_j). \tag{3.3}$$

Correspondingly, (3.2) is now rewritten as

$$P_f(x) = \sum_{j=1}^{N} c_j \phi(x - x_j). \tag{3.4}$$

To express the problem in matrix–vector form, we let $X = \{x_1, \cdots, x_n\}$ be set of pairwise distinct centers and let

$$c = [c_1, \cdots, c_N]^T, \quad f = [f_1, \cdots, f_N]^T, \quad A_{X,\phi} = (\phi\|x_i - x_j\|).$$

Then enforcing the interpolation constraint (3.1) with a P_f in the form of (3.4) leads to

$$A_{X,\phi} c = f. \tag{3.5}$$

The fact, that ϕ is assumed to be strictly positive definite guarantees that the interpolation matrix $A_{X,\phi}$ is invertible. Therefore, $c = A^{-1}_{X,\phi} f$.

In this classical setup, there is no restriction on the distribution of the data sets x_j except for being pairwise distinct. However, distribution of the x_j is an important issue in the so–called multistep interpolation method, i.e., the sequence of data are to be in increasing subsequences. In order to define multistep interpolation, we consider normal multiresolution approximation. For simplicity, this has been described in the following section for 1–D case, since it may be easily generalized to multivariate case with or without some modification, i.e., the modification does not affect the generality of underlying hypothesis in this scheme.

3.2 Multistep Interpolation

To begin with, we are given a smooth curve f. This curve can be parametrized in many ways; often we shall assume it is C^1 continuous, hence we could parametrize it by arc length. It will be more convenient for us, though, to parametrize it by one of the x– or y–coordinates. A piecewise C^1 function can always be broken up into adjacent finite length pieces, possibly overlapping, that can be well parametrized by the x–coordinate (with, say, $|dy/dx| \leq 2$) or by the y–coordinate (with, say, $|dx/dy| \leq 2$); by restricting ourselves to these different pieces separately, and interchanging the names of the two coordinates, we may thus assume that the curve f is parametrized by its x–coordinate, so that

$$f = \{(x, \gamma(x)) : x \in I\},$$

where f is a smooth function and I is an interval, a half–line, or all of $\mathbb{R}$. They can be applied to each piece of the curve individually and yield uniform results for the whole curve if the number of pieces are finite. For convenience, always assume that the definition of γ is extended to all of $\mathbb{R}$. In many cases, we shall assume that Lipschitz continuity with a Lipschitz exponent $0 < \beta \leq 1$.

Given a (possibly finite) sequence x_j of X_j in I, we define $y_j = \gamma(x_j)$. For every j, we compute the two predictor sequences x^*_{j+1} and $y^\star_{j+1}$ using an interpolating stationary linear subdivision scheme S,

$$x^*_{j+1} = Sx_j, \quad y^*_{j+1} = Sy_j.$$

These are in general not related via the function γ , i.e., $y^*_{j+1} \neq \gamma x^*_{j+1}$. In a normal multiresolution, first determine, for every k, the line through the point $(x^*_{j+1,2k+1}, y^*_{j+1,2k+1})$ that is perpendicular to the line connecting $(x_{j,k}, y_{j,k})$ and $(x_{j,k+1}, y_{j,k+1})$; the intersection point of this normal line and the curve of f gives the new point $(x_{j+1,2k+1}, y_{j+1,2k+1})$. The x–coordinate of this new odd–indexed point thus satisfies

$$(x_{j+1,2k+1} - x^*_{j+1,2k+1})(\Delta x_j)_k + (y_{j+1,2k+1} - y^*_{j+1,2k+1})(\Delta y_j)_k = 0; \tag{3.6}$$

the even–indexed points are just taken over from the previous level $x_{j+1,2k} = x_{j,k}$. Let the whole procedure be described by the application of the nonlinear operator N_j to the original sequence

$$x_{j+1} = N_j x_j.$$

Always start out with a strictly increasing sequence x_0, i.e.,$\Delta x_0 > 0$; in order to avoid messy difficulties with the definition of polygonal line. We would like to have $\Delta x_j > 0$, for all j. In general, (3.6) does not always have solutions such that this is true, however, we shall derive conditions on S, x_0, and f to ensure this. In any case, we shall apply the operators N_j only to sequences x_j for which $\Delta x_j > 0$. Also, remark that (3.6) may have several solutions for which $\Delta x_j > 0$. For definiteness assume that there is a rule established which uniquely picks out one of the solutions, should there be many. The rule could for instance be to pick the solution closest to (or furthest away from) the predicted point. When we say that the points on the next finer level are **well defined**, we mean that there exist solutions x_{j+1} with $\Delta x_{j+1} > 0$ satisfying (3.6) and, if there are many such solutions, we implicitly assume that the rule decides which of them to select.

In order to define the convergence we wish to establish, we introduce auxiliary functions γ_j. Each γ_j interpolates linearly the values $y_{j,k}$ at the $x_{j,k}$; if x_j is strictly increasing, this is a well–defined function. Without restriction, also assume that I is the smallest interval containing all points $x_{j,k}$ so that γ_j is defined on the whole of I. The graph of γ_j, the (piecewise linear) curve of f_j, is the normal multiresolution approximation at level j. (Note that f_j depends on f, x_0 and S as well as on j). We will then say that the normal multiresolution approximation f_j converges to f if

$$\|\gamma(x) - \gamma_j(x)\|_{L^\infty(I)} = \sup_{x\in I} |\gamma(x) - \gamma_j(x)|$$

converges to 0 as $j \to \infty$. Now, if $\gamma \in C^\beta$ and $\tilde{\beta} = \min(\beta, 1) > 0$ then

$$\sup_{x_{j,k}\le x\le x_{j,k+1}} |\gamma(x) - \gamma_j(x)| \le \Omega(\tilde{\beta}, \gamma)(\Delta x_j)_k^{\tilde{\beta}}, \tag{3.7}$$

so that

$$\|\gamma - \gamma_j\|_{L^\infty(I)} \le C\ |\Delta x_j|_\infty^{\tilde{\beta}}.$$

The normal multiresolution approximation therefore converges to the desired limit if x_j remains strictly increasing for all j and if $|\Delta x_j|_\infty \to 0$ as $j \to \infty$.

We shall occasionally single out one particular family of interpolating subdivision schemes for use in the prediction step: the so–called Lagrange interpolation subdivision schemes, in which the new odd–indexed points are given the values taken by a polynomial determined by several neighbouring old points. For instance, in the two–point scheme, $u_{j+1,2k+1}$ is given the value at $t = 1/2$ of the linear polynomial that takes the values $u_{j,k}$ at $t = 0$ and $u_{j,k+1}$ at $t = 1$; in other words,

$$u_{j+1,2k+1} = \frac{1}{2}(u_{j,k} + u_{j,k+1}).$$

In the four-point scheme, $u_{j+1,2k+1}$ is given the value at $t = 1/2$ of the cubic that takes the values $u_{j,k-1}$, $u_{j,k}$, $u_{j,k+1}$, and $u_{j,k+2}$ at t = -1, 0, 1, 2, respectively, leading to

$$u_{j+1,2k+1} = \frac{9}{16}(u_{j,k} + u_{j,k+1}) - \frac{1}{16}(u_{j,k-1} + u_{j,k+2}).$$

In general, the $2l$–point scheme gives $u_{j+1,2k+1}$, the value at $t = 1/2$ of the $(2l-1)$–degree polynomial that takes the values $u_{j,k+m}$ at $t = m$ where $m = -l+1, \cdots, l$. We shall denote the $2l$–point scheme by S_{2l}. In particular, the two–and four–point schemes will be denoted by S_2 and S_4:

$$(S_2u_j)_{2k+1} := \frac{1}{2}(u_{j,k} + u_{j,k+1}).(S_4u_j)_{2k+1} := \frac{9}{16}(u_{j,k} + u_{j,k+1}) - \frac{1}{16}(u_{j,k-1} + u_{j,k+2}).$$

Since these are all interpolating schemes we have, of course, $(S_2u_j)_{2k} = (S_4u_j)_{2k} = (S_{2l}u_j)_{2k} = u_{j,k}$. When the prediction step is computed by means of S_2, i.e.,

$$x^*_{j+1} = S_2x_j, \quad y^*_{j+1} = S_2y_j,$$

it turns out that the analysis of normal multiresolution approximation is especially simple.

3.2.1 An Iterative Procedure for Interpolation

The above procedure is analogous to the version of Newton's method for polynomial interpolation in one dimension. Let the interpolation condition be $S_{f,X}(x_j) = f(x_j)$, $j = 1, 2, \cdots, n$ where x_j, $j = 1, 2, \cdots, n$ are points of X that are all different, and where $S_{f,X}$ is now required to be polynomial of degree at most $n - l$ from $\mathbb{R}$ to $\mathbb{R}$. Then the Lagrange functions $\chi_n(x) = 1$, $x \in \mathbb{R}$, and

$$\chi_k(x) = \prod_{j=k+1}^{n} \frac{x - x_i}{x_k - x_i}, \quad x \in \mathbb{R}, \quad k = 1, 2, \cdots, n-1, \tag{3.8}$$

and we write $S_{f,X}$ in the form

$$S_{f,X}(x) = \sum_{k=1}^{n} \mu_k \chi_k(x), \quad x \in \mathbb{R}. \tag{3.9}$$

It follows that the partial sum

$$S_{f,X}(x) = \sum_{k=l}^{n} \mu_k \chi_k(x), \quad x \in \mathbb{R}, \tag{3.10}$$

is unique polynomial of degree at most $n-l$ that interpolates the last $n-l$ data, where l is any integer in $[1,n]$. Further, μ_l is the coefficient of x^{n-l} in S_l divided by coefficient of x^{n-l} in χ_l. Thus, it is possible to calculate the parameters μ_k, $k = 1, \cdots, n$, of formula (3.9).

Specifically, for radial basis functions interpolation in $d \geq 1$ dimension to the data $f(x_j) = f_j$, $j = 1, \cdots, n$, analogue of expression of (3.8) can be represented as

$$\chi_k(x) = \sum_{j=k}^{n} \lambda_{k,j} \phi(\|x - x_k\|) + p_k(x), \quad x \in \mathbb{R}^d, \tag{3.11}$$

whose parameters are fixed by the condition

$$\chi_k(x_j) = \delta_{j,k}, \quad j = k, k+1, \cdots, n \tag{3.12}$$

$$\sum_{j=k}^{n} \lambda_{k,j} = 0 \quad and \quad \sum_{j=k}^{n} \lambda_{k,j} x_j = 0, \tag{3.13}$$

where ϕ is a radial basis function and where p_k is polynomial from $\mathbb{R}^d$ to $\mathbb{R}$ of degree at most $d-1$. Thus, the Lagrange function χ_k from a space

$$\overline{span\{\phi(\cdot - x_1), \cdots, \phi(\cdot - x_n)\}}$$

for $X := \{x_1, \cdots, x_n\}$. Hence, the normal interpolating points are interpolated by radial basis functions. Thus, the problem is set to

$$S_k(x) = \sum_{j=1}^{n} \mu_j \varphi(x - x_j) + \sum_{l=1}^{N} \beta_l p_l(x), \quad for\ x \in \mathbb{R}. \tag{3.14}$$

Now, we have to specify the condition on which the radial basis function has a unique representation for the above normal interpolating functions.

3.3 Interpolating Basis Function

Now, consider interpolation of real-valued functions f defined on a set $\Omega \subset \mathbb{R}^d$, $d \geq 1$. These functions are evaluated on a set $X := \{x_1, \cdots, x_{N_X}\}$, $(N_X \geq 1)$ of pairwise distinct points $x_1, \cdots, x_{N_X}$ in Ω. If $N_X \geq 2$, $d \geq 1$ and $\Omega \subset \mathbb{R}^d$ are given.

It is well known that there is no d-dimensional space of continuous functions on Ω that contains a **unique interpolant** for every f and every set $X := \{x_1, \cdots, x_{N_X}\}$ consisting of N_X data points. Thus, the family of interpolants must necessarily depend on X. This can easily be achieved by using translates $\Phi(x - x_j)$ of a single continuous real-valued function Φ defined on $\mathbb{R}^d$.

Further simplification is obtained by letting Φ be radially symmetric, i.e.:

$$\Phi(x) := \phi(\|x\|_2), \tag{3.15}$$

with a continuous real-valued function ϕ on $\mathbb{R}^d$ and the Euclidean norm $\|.\|_2$.

Interpolants $S_{f,X}$ to f can then be constructed via the representation

$$S_{f,X} = \sum_{j=1}^{N_X} \alpha_j \Phi(x - x_j), \tag{3.16}$$

where the coefficients $\alpha_1, \cdots, \alpha_{N_X} \in \mathbb{R}$ solve the linear system

$$f(x_k) = \sum_{j=1}^{N_X} \alpha_j \Phi(x_k - x_j), \quad 1 \leq k \leq N_X,$$

provided that the symmetric $N_X \times N_X$ matrix

$$A_{X,\Phi} := \begin{pmatrix} \Phi(x_1 - x_1) & \cdots & \Phi(x_1 - x_{N_X}) \\ \vdots & \ddots & \vdots \\ \Phi(x_{N_X} - x_1) & \cdots & \Phi(x_{N_X} - x_{N_X}) \end{pmatrix}$$

is nonsingular. This is the simplest form of radial basis function interpolation, but for a variety of choices of Φ, it is necessary to add polynomials to the interpolant (3.16).

Let P_q^d denote the space of d-variate polynomials of order not exceeding q, and let the polynomials $p_1, \cdots, p_Q$ be a basis of P_q^d in $\mathbb{R}^d$. The Q additional degrees of freedom of the extended representation

$$S_{f,X} = \sum_{j=1}^{N_X} \alpha_j \Phi(x - x_j) + \sum_{l=1}^{Q} \beta_l p_l(x) \tag{3.17}$$

are compensated by the Q additional equations

$$\sum_{j=1}^{N_X} \alpha_j p_l(x_j) = 0, \quad 1 \le l \le Q. \tag{3.18}$$

With the matrix

$$P_X^T := \begin{pmatrix} p_1(x_1) & \cdots & p_1(x_{N_X}) \\ \vdots & \ddots & \vdots \\ p_Q(x_1) & \cdots & p_Q(x_{N_X}) \end{pmatrix},$$

we can write the interpolation conditions

$$f(x_k) = \sum_{j=1}^{N_X} \alpha_j \Phi(x_k - x_j) + \sum_{l=1}^{Q} \beta_l p_l(x_k), \quad 1 \le k \le N_X,$$

together with (3.18) as a linear system

$$\begin{pmatrix} A_{\Phi,X} & P_X \\ P_X^T & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} f_X \\ 0 \end{pmatrix}, \tag{3.19}$$

where the data from f form a vector $f_X := (f(x_1), \cdots, f(x_{N_X}))^T$. Solveability of this system depends on two conditions. First, the matrix $A_{\Phi,X}$ should be nonsingular on the vectors α satisfying (3.18). Second, polynomials in P_q^d should be uniquely determined by their values on X, i.e., $p \in P_q^d$ satisfies $p(x_i) = 0$, for all $x_i \in X$, then $p = 0$.

The space

$$V_N = \overline{span\{\Phi(\cdot - k) : k \in \mathbb{Z}^d\}}$$

form a multiresolution analysis for $\mathcal{F}(\Omega)$ and contain Π_d, where $\mathcal{F}$ is some known function spaces, such as Sobolev spaces.

Given $\Omega \subset \mathbb{R}^d$, it is to construct suitable subspaces of V_j whose restriction $V_j(\Omega)$ to Ω from a multiresolution of $\mathcal{F}(\Omega)$:

$$V_0(\Omega) \subset V_1(\Omega) \subset \cdots \subset \mathcal{F}(\Omega), \quad \overline{\bigcup_{j=0}^{\infty} V_j(\Omega)} = \mathcal{F}(\Omega),$$

and contain the space $\Pi_N(\Omega)$ of all polynomials of coordinate degree N of Ω.

The discussion of the first condition is simplified if non–singularity is replaced by positive definiteness of the basis function. Moreover, whence the interpolating basis function is multiquadric or compactly supported $\sum_{l=1}^{Q} \beta_l p_l(x_k) = 0$. It further simplifies the representation of a function being interpolated.

Definition 3.3.1 *A function $\Phi : \mathbb{R}^d \to \mathbb{R}$ with $\Phi(-x) = \Phi(x)$ is conditionally positive definite of order q on $\mathbb{R}^d$, if for all sets $X = \{x_1, \cdots, x_{N_X}\} \subset \mathbb{R}^d$ with N_X distinct points and all vectors $\alpha := \{\alpha_1, \cdots, \alpha_{N_X}\} \in \mathbb{R}^{N_X}$ with (3.18) the quadratic form*

$$\sum_{j,k=1}^{N_X} \Phi(x_j - x_k) = 0$$

attains nonnegative values and vanishes only if $\alpha = 0$.

In a fundamental paper Micchelli (1986) related the conditional positive definiteness of functions Φ of the form (3.15) to complete monotonicity of derivatives of all most everywhere, and this technique allows to prove conditional positive definiteness for a variety of radial basis functions φ, such as thin plate splines, where $\varphi(x) = x^2 \log(x)$, for all $x \in \mathbb{R}^d$, which are particularly suited for interpolation from planar scattered data. Further commonly used radial basis functions are given by the Gaussians, $\varphi(x) = \exp(-x^2)$, the multiquadric $\varphi(x) = (c^2 + x^2)^\beta$, $\beta > 0$ and $\beta \notin \mathbb{N}$ and the inverse multiquadric, $\varphi(x) = (c^2 + x^2)^\beta$, $\beta < 0$, for all $x \in \mathbb{R}^d$, where c is a positive constant.

Remark 3.3.1 *The positive definiteness of Φ guarantees that all possible interpolation problems posses a unique solution and this then justify referring to Φ as a basis function. Thus, the interpolation equation (3.16) define the coefficients $\{\alpha_j : j = 0, 1, \cdots, N\}$ uniquely for any given right–hand side of $\{f_j : j = 0, 1, \cdots, N\}$. The definition of the basic function Φ often involves with parameters λ and β defined as above, e.g., the multiquadric*

$$\phi_\lambda(x) := (x^2 + \lambda^2)^{\beta/2}, \quad for\ all\ x \in \mathbb{R}^d,$$

whose properties are well understood, both theoretically as well as practically. One of the reasons for this particular function is that desire to use the $\lambda > 0$ as a tension parameter. These shape parameters can be used to control the flatness of ϕ, and finding a good values for these parameters is a major issue of data approximation (see e.g., Fornberg and Zuev

(2007), Frank (1982)). The function $S_{f,X}$ based on (3.17) exactly interpolates the given data, when N is large, solving this (often dense and ill–conditioned) linear system can be rather time consuming.

Curious enough, each conditionally positive definite function Φ does not only define an interpolation method, but also defines an inner-product space F_Φ of functions. In the following section, it is described the construction of such a space V_j, for $j = 0, 1, \cdots$, and introduced the exponentially decaying positive definite radial basis functions that generate the Sobolev spaces W_2^k.

Remark 3.3.2 *Although, this study concentrated on the interpolatory binary subdivision schemes using the multiquadric, the proposed approach can also be applied to any basis function ϕ whose Fourier transform $\hat{\phi}$ coincides on $\mathbb{R}\backslash 0$ with some continuous function while having a certain type of singularity (necessarily of a finite order) at the origin, i.e., $\hat{\phi}$ is of the form $|\cdot|_n\hat{\phi} = F > 0$ with $n \geq 0$ and $F \in L_\infty(\mathbb{R})$. For instance, the Gaussian function $\phi(x) := e^{-cx^2}$, $c > 0$, and inverse multiquadric function $\phi(x) := (x^2 + \lambda^2)^{-1/2}$, $\lambda > 0$, can be candidates.*

3.4 Multiscale Approximation

In particular, we have increasing sequence of x_j of X_j with nestedness. Therefore, hierarchical method which starts with a decomposition of X into nested sequences such that

$$X_1 \subset X_2 \subset \cdots \subset X_{M-1} \subset X_M = X. \tag{3.20}$$

This allows the interpolation problem to be broken–up into M steps. The decomposition strategy make use of Normal Subdivision (NS) triangulation and is designed so that the density of the points in each X_j is as uniform as possible and increases smoothly as j increases.

In this section, we shall use $(x, \gamma_k(x))$ to denote the points in $\mathbb{R}^d$. Also, assume that the boundary of Ω is a simple closed curve Γ which is the union of curves Γ_k, $k = 1, 2, \cdots, m$, with the following properties: Γ_k has endpoints p_{k-1} and p_k when traversed in a clockwise direction. For even k, Γ_k can be parameterized by $(x, \gamma_k(x))$, $x \in I_k$, with I_k, an interval with endpoints x_{k-1} and x_k and γ_k is in $Lip_M 1$. Recall that $Lip_M 1$ is the set of all continuous univariate functions g which satisfy $\|g^{'}\| \leq M$. We shall assume (without loss of generality) that $M \geq 1$. When k is odd, Γ_k can be parameterized by $(\gamma_k(y), y)$, $y \in I_k$, with I_k be an interval with endpoints y_{k-1} and y_k and γ_k is in $Lip_M 1$.

Given this setting, it is reasonable to infer the convergence of such methods. For this, it is assumed that the analysis is performed in a bounded open region of $\mathbb{R}^d$, and the data values $\{f_j : j = 1, 2, \cdots\}$ come from some continuous function $F : \mathbb{R}^d \to \mathbb{R}$ with

$$F(x_j) = f_j \quad j = 1, 2 \cdots , .$$

Then if $S_m(x)$ is the function which interpolates the data at $\{x_j : j = 1, 2, \cdots , m\}$, and if $\{x_j : j = 1, 2, \cdots , \}$ become dense, we ask whether there is a (maybe smaller) bounded open region of $\mathbb{R}^d$ on which

$$|S_m(x) - F(x)| \to 0 \quad as\ m \to \infty. \tag{3.21}$$

A more fundamental question, however, is whether this property holds, not for the function which interpolates, but for the best approximation from the linear space. So, given $\{x_j : j = 1, 2 \cdots , \}$ which become dense, do there exist a sequence of functions $t_m(x)$, such that

$$t_m(x) = \sum_{i=1}^{m} \lambda_i^m \varphi(\|x - x_i\|) + \sum_{j=1}^{n} \mu_j^m p_j(x),$$

(where it is no longer necessary that $t_m(x_i) = f_i$, for $i = 1, \cdots , m$) and some bounded open domain on which

$$|t_m(x) - F(x)| \to 0 \quad as \quad m \to \infty.$$

By Jackson (1988), sufficient conditions on the linear space for the above result to hold by assuming φ satisfies the homogeneity condition, and assuming that there exist $\{\mu_i \in \mathbb{R} : i = 1, 2, \cdots , m\}$ and $\{x_i \in \mathbb{R} : i = 1, \cdots , m\}$ such that the function

$$h(x) = \sum_{i=1}^{m} \mu_i \varphi(\|x - x_i\|) \tag{3.22}$$

has the following properties:

$$\int_{\mathbb{R}^d} |h(x)| dx < \infty, \tag{3.23}$$

and

$$\int_{\mathbb{R}^d} h(x)dx \neq 0. \tag{3.24}$$

We briefly review the results stated in Jackson (1988).

Theorem 3.4.1 *Suppose that φ is homogeneous, then given a function $h(x)$ defined in (3.22) and satisfies (3.23) and (3.24). In a bounded open domain D with closure$(D) \subset \tilde{D}$ a continuous function F : closure$(\tilde{D}) \to \mathbb{R}$ and a sequence of points $\{z_i : i = 1, 2, \cdots ,\}$ which become dense in $\tilde{D}$. Suppose there exist N and $\{\lambda_i : i = 1, 2, \cdots , N\}$ such that for all $x \in D$, then we have*

$$|F(x) - \sum_{i=1}^{N} \lambda_i \varphi(\|x - x_i\|)| < \epsilon \quad for\ \epsilon > 0.$$

The above theorem states the general (sufficient) condition for local uniform convergence of the interpolating scheme. As a special case, existence of such function $\varphi(r) = r$ is considered when d is odd. This can be generalized to cover the cases $\varphi(x) = \sqrt{x^2 + c^2}$ and $\varphi(x) = 1/\sqrt{x^2 + c^2}$.

Theorem 3.4.2 *In $\mathbb{R}^d$, $d \geq 1$ and d is odd, then there exist a function $h(x)$ as defined in (3.22) and satisfying (3.23) and (3.24) with $\varphi(x) = \sqrt{x^2 + c^2}$ uniformly converges locally.*

Theorem 3.4.3 *In $\mathbb{R}^{2d+1}$, $d \geq 1$ there exist function $h(x)$ as defined in (3.22) and satisfying (3.23) and (3.24) with $\varphi(x) = 1/\sqrt{x^2 + c^2}$ uniformly converges locally.*

Proofs of above two theorems can be found in Buhmann (1990).

Both multiquadric and inverse–multiquadric are, however, not integrable. But, each φ can be considered as tempered distributions or generalized functions, and their Fourier transform in the sense of generalized functions are for $r = \|x\|$,

$$\widehat{\varphi}(r) = -\pi^{-1}(2\pi c/r)^{(n+1)/2} K_{(n+1)/2}(cr), \quad r > 0, \tag{3.25}$$

when $\varphi(r) = \sqrt{r^2 + c^2}$ and

$$\widehat{\varphi}(r) = 2(2\pi c/r)^{(n-1)/2} K_{(n-1)/2}(cr), \quad r > 0, \tag{3.26}$$

when $\varphi(r) = 1/\sqrt{r^2 + c^2}$. Here $\{K_j(z) : z > 0\}$ for $j \geq 0$ are modified Bessel functions which are positive and smooth in $\mathbb{R}^+$, have a pole at origin and decay exponentially.

3.5 Space Spanned by Basis Function

In this section we study the approximation properties of this interpolation process in case of denser and denser sets of centers X. To this end we introduce the necessary results about native spaces, fill distance and numerically accessible Power functions. The power functions that arise from interpolation with s radial basis function Φ on a space $\mathcal{F}_\Phi$ defined by different radial basis functions Φ.

To estimate the error on any interpolation or approximation, we have to assume that the (unknown) function, we are interested in, comes from a known space. In the theory of radial basis function interpolation this space is in general the native Hilbert space of the underlying basis function. The native space $\mathcal{F}_\Phi$ consists of all distributions $f \in S'$ that have a generalized Fourier transform $\hat{f}$ that satisfies $\frac{\hat{f}}{\sqrt{\hat{\Phi}}} \in L_2(\mathbb{R}^d)$. In particular, if Φ is positive definite with $\Phi \in L_1(\mathbb{R}^d)$ then

$$\mathcal{F}_\Phi = \{f \in L_2(\mathbb{R}^d) : \frac{\hat{f}}{\sqrt{\hat{\Phi}}} \in L_2(\mathbb{R}^d)\}.$$

The native space $\mathcal{F}_\Phi$ possesses the semi–norm

$$|f|_\Phi^2 = (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{|\hat{f}(\omega)|^2}{\hat{\Phi}(\omega)} d\omega \tag{3.27}$$

with the null space P_m^d. Thus, $|\cdot|_\Phi$ is a norm if Φ is positive definite. In this case $\mathcal{F}_\Phi$ is a Hilbert space. If Φ is conditionally positive definite of order $m > 0$, then the space $\mathcal{F}_\Phi / P_m^d$ is a Hilbert space. For functions $u \in \mathcal{F}_\Phi$ it is possible to bound the error by

$$|f(x) - S_f(x)| \leq P_{X,\Phi} |u|_\Phi, \tag{3.28}$$

with the so-called Power function $P_{X,\Phi}$ defined pointwise as the norm of the error functional. This Power function can be bounded in terms of the local data density given by

$$h_\rho(x) := \sup_{\|y-x\|_2 < \rho} \min_{1 \leq j \leq N} \|y - x_j\|_2, \quad \rho > 0.$$

But, if we restrict ourselves to basis functions having an algebraically decaying (generalized) Fourier transform, we choose $X \subset \Omega$ and to bound the Power function also in terms of the global data density

$$h = h_{X,\Omega} := \sup_{x \in \Omega} \min_{1 \leq j \leq N} \|x - x_j\|_2, \tag{3.29}$$

as long as satisfies a uniform interior cone condition. In this case, the Power function can be bounded via $P_{X,\Phi}(x) \leq CF(h)$.

However, if Φ is the Gaussian then the native space is rather small, since the Fourier transform of any function from the native space must decay faster than the Fourier transform of the Gaussian which is a Gaussian itself. This argument is often used to diminish the importance of native spaces. But, even the native space to Gaussian contains at least all band–limited functions and this space plays an important role in sampling theory, in particular, in Shannon's famous sampling theorem.

3.6 Approximation in Sobolev spaces

It is interesting to turn the investigation into the approximation error between f and the $S_{f,X}$ coming from V_N of $W_2^k(\Omega)$, where V_N is given by

$$V_N := span\{\Phi(\cdot - x_1), \cdots, \Phi(\cdot - x_N)\} + \mathbb{P}_m^d$$

belonging to a special positive definite function $\Phi : \mathbb{R}^d \to \mathbb{R}$ is at least a C^1 function. $\mathbb{P}_m^d$ denotes the space of polynomials of degree less than m and $X = \{x_1, \cdots, x_N\} \subset \Omega$ is a set of pairwise distinct centers. The most interesting case is when Φ is compactly supported and $m = 0$, i.e., no polynomials are added. That is the Interpolating Normal Basis function which is compactly supported. In this case the stiffness matrix

$$a(\Phi(\cdot - x_j), \Phi(\cdot - x_k))$$

is sparse. Moreover, for a radially symmetric and a radial Φ, i.e., $\Phi(x) = \phi(\|x\|_2)$, $x \in \mathbb{R}^d$, with a function $\phi : \mathbb{R}^d \to \mathbb{R}$, most of the entries of the stiffness matrix can be easily computed.

Construction of such basis function is as follows: consider the transform $\varphi(\|\cdot\|) : \mathbb{R}^d \to \mathbb{R}$ to $\widehat{\varphi}(\|\cdot\|) : \mathbb{R}^d \to \mathbb{R}$, the transform would be the function

$$F(x) := \sum_{j \in \mathbb{N}} \mu_j e^{-i(x,j)} \widehat{\varphi}(\|x\|), \quad x \in \mathbb{R}^d. \tag{3.30}$$

Thus, F can be continued into the origin, i.e., $F(0) := \lim_{\|x\| \to 0} F(x)$ is well defined, and $F(x)$ is k–times continuously differentiable at 0 (it is smooth elsewhere in $\mathbb{R}^d$).

Assume that u to be somewhat more regular, say $u \in W_2^k(\Omega)$ with $k > d/2$. Moreover, according to the Lipschitz smoothness of the boundary Ω there is a continuous extension mapping $E : W_2^k(\Omega) \to W_2^k(\mathbb{R}^d)$, and denote the extended function $Eu \in W_2^k(\mathbb{R}^d)$ by again u. This allows to use the theory of radial basis functions to identify $W_2^k(\mathbb{R}^d)$ with the native space $\mathcal{F}_\Phi$ to a radial basis function $\Phi \in L_1(\mathbb{R}^d)$ with Fourier transform Φ having the property

$$c_1(1 + \|\cdot\|)^s \leq \Phi(\cdot) \leq c_2(1 + \|\cdot\|)^s, \tag{3.31}$$

where $s = d/2 + k + 1$ and c_1 and c_2 are some positive constants. On account of norm equivalence we have

$$|f(x) - S_f(x)| \leq C\|f\|_{W_2^k(\mathbb{R}^d)} P_{X,\Phi}(x). \tag{3.32}$$

In view of the function $F(x)$, we have constructed above, the power function $P_{X,\Phi}(x)$ can be bounded from above in the following manner. There exists a h_1 such that for all X with $h \leq h_1$ and all $x \in \Omega$ the estimate

$$|P_{X,\Phi}(x)| \leq Ch^{k+1} \tag{3.33}$$

is valid. Here C denotes a positive constant independent of x and X. We say that a set Ω has the cone property if and only if there exists a $\theta > 0$ and $r > 0$ such that for all $t \in \Omega$ a unit vector $\zeta(t)$ exists such that the cone

$$C(t) := \{t + \lambda\eta : \eta \in \mathbb{R}^d,\ \|\eta\|_2 = 1,\ \eta^T\zeta(t) \geq cos\theta,\ 0 \leq \lambda \leq r\}$$

is contained in Ω. It is obvious normal subdivision scheme satisfies cone condition then we have the following theorem.

Theorem 3.6.1 *Let $s = d/2 + k + 1$ and Φ satisfies (3.31). Let Ω be a bounded open subset of $\mathbb{R}^d$ having Lipschitz boundary $\partial\Omega$. Then there exist constants h_0 and C such that for every $f \in W_2^k(\mathbb{R}^d)$ the interpolant $S_{f,X}$ on $X = \{x_1, \cdots x_N\} \subset \Omega$ satisfies*

$$\|f - S_f\| \leq C\|f\|_{W_2^k(\Omega)} h^{k+1}, \tag{3.34}$$

if $h \leq h_0$ with h defined as

$$h := \sup_{x \in \Omega} \min_{1 \leq j \leq N} \|x - x_j\|. \tag{3.35}$$

Thus, the interpolation with Φ provides approximation of order $k + 1$.

3.7 Summary

In this chapter, a sufficient condition for interpolating basis function of normal multiresolution is established with multiquadric function. Moreover, its multiresolution properties are established with the aid of uniform cone condition in a bounded domain. The remarkable fact is that the interpolating multiscale approximation with a basis function of suitable translations and scaling.

It is clear from the above discussions that the normal multiresolution approximation can be represented with interpolating basis function which is Hardy's multiquadric. But, it does not play any positive role with the wavelet concepts, since the basis function is not compact. Then the following chapter develop an quasi–interpolating basis function with the compact support, with the principle of quasi–interpolating wavelet transform concepts on real line. The basis function is constructed with the second order divided difference of multiquadric functions.

Chapter 4

Quasi–Interpolating Wavelets

4.1 Interpolation Using Multiquadric Functions

Spaces spanned by finitely or countably many translates of one or several functions play an important role in spline theory, radial basis function theory, sampling theory, and wavelet theory. Spline theory stresses the case when the generating functions are compactly supported, while sampling theory singles out the case when the spectrum (i.e., the support of the Fourier transform) of the generating functions is compact. This fact has been known for a long time and usually denoted Shannon's Sampling Theorem. If a function f is not band–limited but has an absolutely integrable Fourier transform $\hat{f}$, then $I_W f$ converges to f, where $I_W f$ represents Whittaker's cardinal series (Sickel, 1992).

Radial basis functions are known to be useful and accurate to the approximation of functions. The key idea is to approximate from a space spanned by translate of single function φ, usually a global support, where the translate take the form $\varphi(|\cdot - x_k|)$ and the x_k's are given centers in a finite set X. They were first introduced in interpolating schemes but, because of their high quality approximation, they are used for many different approximation tools. Radial functions methods are easy to implement. Among all radial basis functions currently in use, the multiquadric radial function $\varphi(r) = \sqrt{r^2 + \lambda^2}$ is probably the best understood both theoretically and from a practical point of view, and also it is the one most frequently used, partly by virtue of the variable real parameter λ.

Moreover, multiquadric function gives exact representation with normal multiresolution approximation meshes.

The best possible choice of spaces $S(\varphi)$ are spanned by translation of multiquadric function with knot spacing $2^{-j}k$, i.e., $\sqrt{(\cdot - 2^{-j}k)^2 + \lambda^2}$, $k \in \mathbb{Z}$, then the known theory of radial function provides essentially the same approximation order as splines and as well as infinite differentiability. The generating functions in the univariate multiquadric setting with scattered center are described by Buhmann (1990) and Beatson and Powell (1992). The theory of prewavelets on non–equally spaced data is described in Buhmann (1994, 1995), in particular, the multiquadric functions and allowing the centers to be scattered. A very general account of generating prewavelets from shift–invariant spaces (including radial function spaces with gridded centers) is given by de Boor et al. (1993), but their technique cannot be applied to scattered data.

The main concern of this dissertation is that constructing compactly supported quasi interpolating basis function ϕ which interpolate the given function f. The basis function which we call it as Quasi–Interpolating Wavelet, i.e., a space $S(\phi)$ which is the topological span of the basis functions. In this case, the underlying space $S(\phi)$ is meant for decomposition of functions from some known spaces, such as Besov spaces. In radial basis function theory, neither of these is assumed, and, instead, the computational simplicity as well as the positive definiteness (i.e., the positivity of the Fourier transform) of the generating functions is preferred. Moreover, the radial basis function does not play any positive role with wavelets, since they are not compactly supported function. In order to circumviate this problem we have to look for such functions which are spanned by these multiquadric functions.

4.2 Abstract Setting

A Schauder basis of a Banach space $(X, \|\cdot\|)$ is any sequence $\{x_n\} \subset X$ such that each $x \in X$ has an expansion

$$x = \sum_{n=0}^{\infty} a_n(x) x_n,$$

convergent in the norm $\|\cdot\|$ with uniquely determined functional $a_n(x)$. According to Banach–Steinhans theorem the partial sums

$$S_n x = \sum_{i=0}^{n} a_i(x) x_i$$

are uniformly bounded as linear operators on X and the functionals a_i are continuous. It should be noted that for the interpolating systems with nodes at $\{x_i\}$, the linear functional $a_i(f)$, for $f \in C(I)$ are necessarily of the form,

$$a_i(f) = \begin{cases} f(x_i) & \text{for } i = 0 \\ f(x_i) - S_{n-1} f(x_i) & \text{for } i \geq 1 \end{cases},$$

and therefore $a_i(f)$ is a linear combination of the functional $\delta_i f = f(x_i)$, $i = 0, 1, \cdots, n$ for $n \geq 0$.

Let c be some string of data $c(k)$, $k \in I$, where I is some (finite or possibly infinite) index set. These data could represent gray scale values of a digital image, statistical noisy data, or some control points in some curve or surface representation, or approximation solutions of some discreterized operator equation. The common ground for these rather different interpretation is that these data could be viewed as coefficient of some expansion

$$f = \sum_{k \in I} c(k) \varphi_k, \tag{4.1}$$

where the φ_k (typically scaler–valued) functions defined on some domain (or manifold) Ω (which is topologically equivalent to some bounded or unbounded domain) in $\mathbb{R}^d$. As a simple example, one could take φ_k as interpolating (wavelet) basis function relative to some knot sequence in an interval Ω. When each $c(k)$ is a point in $\mathbb{R}^d$ say, f represents a space curve. The $c(k)$ then covey geometrical information on the curve or, more precisely, on the location of the points $f(x)$, $x \in \Omega$. It is well known that this kind of information can drawn from the data $c(k)$ under much more general circumstances, namely when the φ_k have good localization properties and sum to one. Note that this localization represents the resolution of the underlying object only with respect to a single scale.

However, in many application it is important to extract or exploit information on the data that could be associated with scales ranging from a very coarse to very fine level, e.g., image processing, noise removal etc. Multiscale representation of data usually convey accurate information about the smoothness or regularity of the underlying continuous object. The central ingredient of this process is an appropriate transformation T which is a bounded linear functional relating the fine scale data c to their multiscale representation d

$$c = Td. \tag{4.2}$$

In this case the transformation T is multiresolution approximation. This is clearly an quasi–interpolating transformation. Hence, if we consider the transformation in above settings then there must be quasi–interpolating basis functions which satisfy the properties in Definition 4.2.1.

The following definition is derived from Donoho (1992) with the property of quasi–interpolation.

Definition 4.2.1 *A quasi–interpolating wavelet is a* ***father function*** φ*, satisfying the following conditions.*

IW1. Quasi–Interpolation. φ *as the unique piecewise continuous functions relative to an interval* I_j *satisfying*

$$\varphi_{j,k}(m) = \delta_{m,k}, \quad m, k \in I_j, \tag{4.3}$$

where δ *is the Kronecker sequence.*

IW2. Nestedness. For some $I_j \subset I_{j+1}$*, the basis function* $\varphi_{j,k}$*, satisfy*

$$\varphi_{j,k} = \sum_{m \in I_{j+1}} \varphi_{j,k}(m)\varphi_{j+1,k}. \tag{4.4}$$

IW3. Polynomial span. Interpolates the values of a given function f*, at the given distinct interpolation points of* $\mathbb{R}$ *with* 1*–unisolvent set.*

IW4. Regularity. For some real $0 < \beta \leq 1$*,* φ *is Lipschitz continuous of order* β*.*

IW5. Localization. φ *and its derivative decay rapidly*

$$|\varphi^m(x)| \leq C(1 + |x|)^{-1-\varepsilon}, \quad x \in \mathbb{R}, \quad \varepsilon > 0, \quad m = 0, 1, 2, \cdots . \tag{4.5}$$

IW6. Let $\mathcal{F}$ be the space of collection of uniformly continuous functions. The collection $\mathcal{F}$ of restrictions f_Ω of sums $f = \sum_\alpha c(\alpha)\varphi(\cdot - \alpha)$ is finite dimensional.

By the last property, we have norm equivalence between $L_p(\mathbb{R})$ and $L_\infty(\mathbb{R})$ norms with constants such that

$$N_p(\mathcal{F}) = \frac{\sup_{f\in\mathcal{F}} \|f\|_{L_\infty}}{\|f\|_{L_p}} < \infty, \quad 0 < p \leq \infty.$$

4.3 Multilevel Interpolating Operators

In this section, a more general approach of employing the concepts of quasi–interpolating wavelets using radial basis function is developed. Consider radial function, such as multiquadric function, interpolate the normal multiresolution meshes by its translates. It is clear that wavelet coefficients are determined only in compact neighborhood. It is the characteristic of the quasi–interpolation. Thus, we use the interpolation to characterize the quasi–interpolating wavelet transform.

Recall from Chapter 3 that for normal multiresolution approximation at the resolution level j, there are given points of increasing subsequences, $\{x_k : k \in \mathbb{Z}^d\}$. Assume that these points are equally spaced with distance $2^{-j}k$. They are interpolated with the multiquadric with some non–zero coefficients $\{d_k\}_{k\in\mathbb{Z}^d}$. Here and sequel the notation $2^{-j}k \in \mathbb{Z}^d$ is to denote the points in $\Omega \subset \mathbb{R}^d$ at a resolution level $j \in \mathbb{N}$, for $k = 1, 2, \cdots$. Hence, the interpolation at a level j is given by

$$I_j f = \sum_k d_k \varphi(\|2^j \cdot -k\|), \quad d \geq 1,$$

is well defined and agrees with f on all points at the resolution level j. Here $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^d$.

Let σ_j, for $j > 0$, denote the scaling operator defined by $(\sigma_j f)(x) = f(2^{-j}x)$, $x \in \mathbb{R}^d$. Throughout this section $n = n(\varphi)$ will denote the approximation degree of the given compactly supported function ϕ, that is, n is the largest integer for which the distance (using and $L_p(\mathbb{R})$ norm) of any compactly supported, sufficiently smooth function to the scaled space $S_j = \{\sigma_j g : g \in S(\varphi)\}$ is of order $O(h^{n+1})$. We assume that $n \geq 0$. Hence, the above interpolant span scaled space

$$S(\varphi) := span\{\varphi(2^j \cdot -k) : 2^{-j}k \in \mathbb{Z}^d\}.$$

It is clear from the definition, $S(\varphi)$ defines a principal shift–invariant space, but the topology used in the definition of the space is not defined. The topology is determined by the limit $\sum_k \varphi(2^j \cdot -k)c(k)$, where $c(k)$ is the coefficient sequence. In the absence of standard definition for the space $S(\varphi)$ it is chosen the following one.

Definition 4.3.1 *The principal shift–invariant space is the space of all locally bounded functions f for which the sum*

$$I_j f = \sum_k \varphi(2^j \cdot -k)c(k) \quad as\ j \to \infty,$$

is absolutely convergent for every $x \in \mathbb{R}^d$.

Alternatively, there is a quasi–interpolant such that,

$$Q_j f = \sum_k f(2^{-j}k)\phi(2^j \cdot -k),$$

where $\phi : \mathbb{R}^d \to \mathbb{R}$ is of the form

$$\phi(x) = \sum_k \mu(k)\varphi(\|2^j x - k\|), \quad x \in \mathbb{R}^d, \quad k \in \mathbb{N} \subset \mathbb{Z}^d,$$

which reproduce polynomials $p \in \Pi_d$ such that $Qp = p$, in odd–dimensional Euclidean space $(\mathbb{R}^d, \|\cdot\|)$ of degree d. In case of the inverse multiquadric, $d \geq 3$ is odd, which reproduce polynomials, $p \in \Pi_d$ such that $Qp = p$, of degree d in Euclidean space $(\mathbb{R}^d, \|\cdot\|)$. This has been extensively studied in Buhmann (1990).

It is concerned with the definition of a linear operator Q from $L_\infty \cap C(\mathbb{R}^d)$ into itself with compactly supported and sufficiently smooth functions in $S(\varphi)$

$$Q : f \mapsto \sum_k \psi(\cdot - k)(\Lambda f)(k), \tag{4.6}$$

where Λf is the linear functional and ψ is of the form

$$\psi(x) = \sum_k \mu(k)\varphi(\|x - 2^{-j}k\|), \quad x \in \mathbb{R}^d, \quad k \in \mathbb{N} \subset \mathbb{Z}^d,$$

for some $m_\mu > d$,

$$|\mu(k)| = O(|k|^{-m_\mu}) \quad as\ |k| \to \infty, \tag{4.7}$$

with $f \mapsto \{(\Lambda f)(k)\}$ is some linear assignment. At a minimum, one assumes that Λ is a bounded map from $C(\mathbb{R}^d)$ (equipped with the uniform norm) to $\ell_\infty(\mathbb{Z}^d)$. In such case, the following regularity condition of ψ must be satisfied, for some $m_\psi > d$,

$$|\psi(x)| = O(|x|^{-m_\psi}), \quad as\ |x| \to \infty, \tag{4.8}$$

which guarantees the sum in (4.6) to converges uniformly on compact sets (Dyn et al., 2002). It is very convenient to assume further that Λ commutes with integer shifts, i.e., that $\Lambda f(\alpha) = \Lambda f(\cdot + \alpha)(0)$.

For this purpose, it is important to assume slightly more: the linear operator is a local linear operator to mean a bounded linear operator, say L, whose domain and the range are function spaces on $\mathbb{R}^d$, $d \geq 0$, and which has the property that $(Lf)(x)$ depends only on the values of f in a compact neighborhood $x + K$ of x, where K is a compact subset of $\mathbb{R}^d$ independent of x. Likewise, a local linear functional, say λ, (defined to be later) has the property that λf depends only on the values of f in a compact neighborhood of the origin.

Localization is done with the aid of finitely supported localizing coefficients μ with the property defined to have subtle decay of ψ at ∞. Then we define the interpolant as:

Definition 4.3.2 *Let φ be a multiquadric function $\varphi(r) = (r^2 + \gamma^2)^{\beta/2}$, where $r = \|x\|$, $x \in \mathbb{R}^d$ and γ is some positive parameter with the tension parameter β, which interpolates the points at $\{x_k : k \in \mathbb{Z}^d\}$. Then we have*

$$\phi = \sum_k \varphi(\cdot - k)\mu(k), \quad j \in \mathbb{N}, \tag{4.9}$$

*where $\mu(k)$ is some localization sequence and the decay rate is higher than the defined above and the function ϕ is a functions in $S(\varphi)$, known to satisfies (at least) the following **boundedness condition**.*

$$\sum_k |\phi(\cdot - k)| \in L_\infty(\mathbb{R}^d). \tag{4.10}$$

With the above definition we have more subtle rate of decay of ϕ at ∞. Hence, we have the following bounded linear operator $T : C(\mathbb{R}^d) \to S(\phi)$ defined by

$$Tf = \sum_k (\lambda f)(\cdot + k)\phi(\cdot - k), \tag{4.11}$$

which satisfies

$$Tp = p, \quad for\ all\ p \in \Pi_n,\ n \leq d, \tag{4.12}$$

where the λ is is a bounded map from $C(\mathbb{R}^d)$ (equipped with the uniform norm) to $\ell_\infty(\mathbb{Z}^d)$ and hence (due to (4.12)) T is well defined.

4.3.1 Construction of Quasi–Interpolating Wavelets

The assumption on wavelet basis function is that the basis function is quasi–interpolating function on some known function spaces, Hence, it shares some features in common with Donoho (1992) construction, i.e., wavelet basis functions are not orthogonal or bi–orthogonal. By way of construction, the wavelet proposed in this research agrees with the Donoho's wavelet, in terms of the regularity, polynomial reproduction property and localization conditions. But, it does not agree with the two–scale relation. Moreover, the construction does not inherit the (orthogonal) complement space as opposed to other settings, i.e., it is assumed that simply the finer scale scaling functions as wavelets and the wavelet coefficients are the linear functional λf.

It is important that the sum $\sum_k \phi(\cdot - k)a(k)$ be well defined for any $a(k)$. Therefore, it is assumed that each operator T is well defined and bounded as a map from ℓ_∞ to $C(\mathbb{R})$, and denote the corresponding norm by $\|T\|$. Some conditions related to the boundedness of $\|T\|$ are recorded in the following proposition whose proof is standard (de Boor and Ron, 1992).

Proposition 4.3.1 *The norm of the operator T is $\|\sum_k \phi(\cdot - k)\|$, hence this operator is bounded if and only if the series $|\sum_k \phi(\cdot - k)|$ is pointwise convergent to a bounded function.*

This proposition implies that $\phi \in L_1(\mathbb{R}^d)$ whenever T is bounded, and, hence, that the

Fourier transform $\hat{\phi}$ of ϕ is a well–defined continuous function. Also, a sufficient condition for the boundedness of L is the integrability of the maximal function $\phi^*(x) := \|\phi\|_{L_\infty(x+c)}$.

As is customary, in what follows for any function f defined on $\mathbb{R}^d$ adopt the notation $f_{j,k}(\cdot) := 2^{jd} f(2^j \cdot -k)$, $j, k \in \mathbb{Z}^d$, where j, k are resolution level and integer translates of the function f respectively. The term quasi–interpolating function refers to a function $\phi \in L_1(\mathbb{R}^d)$ whose translates $\phi(\cdot - k)$, constitute a basis for $L_1(\mathbb{R}^d)$.

The construction of quasi–interpolating functions starts on $\mathbb{R}^d$ with a compactly supported function ϕ decay fast enough at ∞ to make the map

$$\phi *^{'} : c \mapsto \sum_k \phi(\cdot - k) f(k)$$

is well–defined and continuous from $\ell_\infty(\mathbb{R}^d)$ to $L_\infty(\mathbb{R}^d)$. Note that such condition implies that $\phi \in L_1(\mathbb{R}^d)$, and is implied by localization condition which employs the approximation map

$$T : f \mapsto \phi *^{'} \lambda(f),$$

where λ is the local linear functional and $\phi *^{'}$ is the semi–discrete convulsion operator(de Boor and Ron, 1992).

It is mainly concerned with a space S_j which is the topological span of the shifts of one generating function $\phi_j(\cdot - k)$. More precisely, hold a collection $\{\phi_j\}_{j \in \mathbb{N}}$ of real–valued measurable functions defined on $\mathbb{R}^d$, where I is either the open interval $(0, h_0)$, or a discrete subset of such an interval (e.g., $2^{-j}k : j \in \mathbb{N}$) in $\mathbb{Z}^d$. For each j, we look at all linear combinations of

$$\sum_k \phi_j(\cdot - k) a(k), \quad j \in \mathbb{N},$$

for which this sum converges in a certain sense and, denote by S_j the space of all limit functions obtained in this way.

$$S_j = \overline{span\{\phi_j(\cdot - k) : k \in \Gamma_j\}}, \quad j \in \mathbb{N},$$

where S_j is the closure of $S(\varphi)$.

For the following, we require the definition of a Lebesgue point of a function f on $\mathbb{R}^d$. Essentially, it is a point x near which the values of f do not deviate too far on the average from the value $f(x)$, and thus can be considered a generalized continuity point.

Definition 4.3.3 *The point x is a Lebesgue point of the function $f(x)$ on $\mathbb{R}^d$ if f is integrable in some neighborhood of x and*

$$\lim_{\epsilon \to 0} \frac{1}{V(B_\epsilon)} \int_{B_\epsilon} |f(x) - f(x+y)| dy,$$

where B_ϵ denotes the ball of radius ϵ about the origin, and V denotes volume.

Remark 4.3.1 *This set of points has full measure in $\mathbb{R}^d$, i.e., its complement has measure 0, so that, convergence of a series on the Lebesgue set implies almost everywhere convergence. Furthermore, all continuity points are also Lebesgue points. Since the set of continuity points of a function can have measure 0 (as for example in the characteristic function of the rational numbers), the Lebesgue set can clearly in some instances be much larger than the continuity set of a function.*

The question is; when this space is admissible? The construction admits, in particular, all pointwise absolutely convergent sums (with real coefficients d_k), and more precisely everywhere on the Lebesgue set of function being expanded

$$\sum_k d_k \phi_j(\cdot - k)$$

into S_j if they are at most of polynomial growth in x and $\phi \in S(\varphi)$. This is clear from the previous discussions. Hence, we seek $P_j f : \mathcal{F} \to S_j$ which agrees in all points of $2^{-j}k \in \mathbb{Z}^d$. Such interpolant always exist if there is a quasi–interpolating function $\chi_j \in S_j$ which satisfies

$$\chi_j(k) = \delta_{0,k}, \quad j \in \mathbb{N},$$

with sufficient decay at ∞. This is always satisfied with our assumptions that the wavelet defined by the Theorem 4.3.1. Roughly speaking, we call S_j the span of the $2^{-j}k$ translates of ϕ, and this is an exact description of S_j. In case of ϕ is of compact support, a case in which the sum $\sum_k \phi_j(\cdot - k)a(k)$ is locally finite, hence, arbitrary linear combinations are allowed in this sum. Hence, P_j is the special type of linear operator which we call it as (non–orthogonal) projection operator onto the subspace S_j.

The function $\phi_j(\cdot - k)$ is then used to construct a multiresolution analysis consisting of an ascending sequence (S_j), $j \in \mathbb{N}$, of subspace of $\mathcal{F}$. Each S_j, $j \in \mathbb{N}$, is generated by the sequence of functions $\{\phi_j(\cdot - k)\}_k$, $j \in \mathbb{N}$ (by the $2^{-j}k$ shifts of ϕ_j. Moreover, as for some $S_j \subset S_{j+1}$, the basis function $\phi_{j,k}$, satisfying

$$\phi_{j,k} = \sum_{m \in I_{j+1}} \phi_{j,k}(m)\phi_{j+1,k}, \tag{4.13}$$

then we have $S_j \subset S_{j+1}$. The functions ϕ_j satisfy the partition of unity property for each $j \in \mathbb{N}$ and interpolating condition, then ϕ_j is linearly independent over Ω_j, for all $j \in \mathbb{N}$. That is, linear operators $(T_j)_{j\in\mathbb{N}}$ satisfy the multiresolution property with the basis function ϕ_j, for all $j \in \mathbb{N}$.

4.3.2 Interpolating Wavelet Transform

Let the knot–sequence X, with support $[x_k, x_{k+n+1}]$, normalized form of a partition of unity and let $\phi \in L_1(\mathbb{R})$ with $\phi(x) \geq 0$, $x \in \mathbb{R}$,

$$\phi(x) = O(|x|^{-n-1-\varepsilon}), \quad x \to \pm\infty, \quad \varepsilon > 0,$$

$\hat{\phi}(\xi) > 0$, $\xi \in \mathbb{R}$ and $\int_{-\infty}^{\infty} \phi = 1$ be given. We assume the following three decay conditions hold:

$$|\phi(x)| = C(1 + |x|)^{-1-\varepsilon},$$

$$|\phi^m(x)| = C(1 + |x|)^{-1-\varepsilon} \quad x \in \mathbb{R}, \quad 0 \leq m \leq \lfloor R \rfloor,$$

where R is the regularity of the function ϕ, and

$$|\hat{\phi}(\xi)| = C(1 + |\xi|)^{-3-\varepsilon},$$

where C is positive generic constant and $\varepsilon > 0$ is arbitrary. We demand the Fourier transform $\hat{\phi}$ with faster rate of decay and continuous at the origin. The Fourier transform $\hat{\phi}$ is well defined (Buhmann et al., 2002).

Now, we are in the position of defining interpolating wavelet satisfies the Definition 4.2.1, for instance, consider the 2^{nd} order divided difference equation of multiquadric function.

Theorem 4.3.1 *Let $f \in C(\mathbb{R})$ be a function, and assume that the non–decreasing sequence of points $\{x_k\}_{k\in\mathbb{Z}}$ are given equally spaced and the function f, can be interpolated* *with multiquadric function $\varphi(r) = \sqrt{r^2 + c^2}$, where $r = \|x\|$ is the Euclidean norm and c is constant. Define*

$$\phi(x) = \frac{1}{2}\sqrt{(x-1)^2 + C^2} - \sqrt{x^2 + C^2} + \frac{1}{2}\sqrt{(x+1)^2 + C^2}, \quad \textit{for all } x \in \mathbb{R}, \tag{4.14}$$

where C is a positive constant. Then the function ϕ is the quasi–interpolating wavelet basis function, which span the space of $S(\varphi)$.

Proof : The first part is an oblivious result from the discussion of the previous chapter.

Consequently the function ϕ satisfies the interpolating property such that

$$\phi_k = \delta_{0,k}, \quad k \in \mathbb{Z},$$

where $\delta_{0,k}$ is the Kronecker symbol that holds 0 if $k \neq 0$ and 1 if $k = 0$. Hence the first property (IW1) of quasi–interpolating wavelet basis is satisfied.

The second property (IW2) is the direct consequence of the definition of wavelet basis function, which is a triangular function, see remark (4.3.2) below.

The third (IW3) property is observed in the previous chapter.

The fourth property (IW4) is obvious from the definition of normal multiresolution approximation, i.e., basis functions are piecewise linear.

The fifth property (IW5) is seen from the above discussion. Final, property (IW6) is satisfied, since the basis function is compactly supported. Hence the theorem is proved. ∎

Remark 4.3.2 *Triangular function that takes the value one at x_{k+1} and vanishes for $x \leq x_k$ and $x \geq x_{k+2}$. Hence, we may consider the process (normal multiresolution approximation) as triangulation of a function on a compact set $\Omega \subset \mathbb{R}$.*

Lemma 4.3.1 *Let $f \in C(\mathbb{R})$ and let the projection operator $P_j : C(\mathbb{R}) \to S_j$ defined as*

$$P_j f = \sum_k f(x_k)\phi_j(\cdot - k), \quad j \in \mathbb{N}, \tag{4.15}$$

with the function ϕ_j defined by theorem (4.3.1) then the following holds

1. *The polynomial reproducing property such that* $P_j p = p, \quad for\ all\ p \in \Pi_D$.

2. *We have the inclusion* $S_j \subset S_{j+1}$.

3. *If* Π_D *denotes all polynomials of degree* $\leq D$, *then* $\Pi_D \subset S_j$.

4. *The formal sum 4.15 at most of polynomial growth for all continuous function* $f \in C(\mathbb{R})$.

Proof : The property (1) is the direct consequence of semi–discrete convulsion property of the operator $P_j f$ and the property (2) is by the definition of wavelet basis function. In particular the basis function is the linear combination of multiquadric (in odd dimension) gives the property (3).

By the third property(IW3), we have for any $\alpha_i \in \mathbb{R}$, $1 \leq i \leq N$ there exists exactly one polynomial $P \in \Pi_k$ such that $P(a_i) = \alpha_i$, where a_i is the interpolation point i.e., for an integer $D \geq 0$, the collection of formal sum

$$\sum_{\alpha \in \mathbb{Z}^d} c(\alpha)\varphi(\cdot - \alpha), \tag{4.16}$$

contain all polynomial of degree D is well defined. Moreover, 4.16 gives us at most polynomial growth for all continuous functions $f \in C(\mathbb{R})$.

Consequently property (4) by the linear operator T_j defined above, since the operator P_j is the special type of operator of T_j, for all $j \in \mathbb{N}$. ∎

The fourth property clearly shows that the operator P_j is a kind of projection operator, i.e., nonorthogonal projection operator onto their respective basis which satisfy the multiresolution property. Thus, we have

$$S_1 \subset S_2 \subset \cdots \subset S_n \subset \cdots = \mathcal{F}(\mathbb{R}),$$

where $\mathcal{F}(\mathbb{R})$, is some topological vector space defined to be later. In the sequel, the L_p, $1 \leq p \leq \infty$ stability of the functions $\{\phi_{j,k}\}$ is needed. We would address this problem in the following chapter.

With the quasi–interpolating wavelet basis function, define the wavelet transform of $f \in C(\mathbb{R})$ by sampling at different resolution levels. By the above lemma 4.3.1 and with our hypothesis we have the following proposition.

Proposition 4.3.2 *Given an interpolating wavelet ψ, of multiquadric and for $f \in C(\mathbb{R})$, we have sequence of projection operators $\{P_j f(x)\}_{j\in\mathbb{N}}$, are known as multiresolution expansion of f. Then the interpolating wavelet transform is defined as*

$$f = \sum_{j\in\mathbb{N}} \sum_{k} c_{j,k}\phi_{j,k}, \tag{4.17}$$

with the coefficients $c_{j,k}$ by sampling $f \in C(\mathbb{R})$ at the resolution level $2^{-(j+1)}$, and coarser with the infinite sum summoned in the topographic order of the basis function ϕ.

Proof : This is the direct consequence of property (4) of Lemma 4.3.1 and the bounded linear operator defined by (4.11). ∎

More interestingly, the above decomposition is natural and, in particular, it is a detailed correction of $\sum_k c_{j,k}\phi_{j,k}$ at each resolution level $j \in \mathbb{Z}$. Thus, the multiresolution approximation with multiquadric is represented with the compactly supported quasi–interpolating basis function ϕ, known as quasi–interpolating wavelet basis function with the wavelet coefficients of distance apart from the two subsequent resolution level at the interpolating points. Which is similar to normal wavelet coefficients. It is natural to ask convergence of the quasi–interpolating wavelet transform.

The following corollary is the consequence of Theorem 4.2 in (Daubechies et al., 2004).

Corollary 4.3.1 *Let $f \in C(\mathbb{R})$ then we have*

$$\max_k |d_{j,k}| \leq C2^{-j\alpha}, \tag{4.18}$$

where $\alpha \in \mathbb{R}$ is the parameter that depends on the the smoothness of the function being interpolated and C is positive constant independent of j and k. Moreover, with the modulus of continuity $\omega(f,\delta)$ defined by

$$\omega(f,\delta) = \sup_{|h|\leq\delta} \sup_{x} |f(x+h) - f(x)|,$$

we have

$$\omega(f, 2^{-j}) \leq M 2^{-j\alpha}, \tag{4.19}$$

where $M > 0$ is depend on the smoothness of the function.

For multiresolution analysis it is necessary to prove the uniform convergence property.

Proposition 4.3.3 *Let $f \in C(\mathbb{R})$, then we have $\|f - P_j f\| = 0$ as $j \to \infty$.*

Proof : Let $\{g_j\}_{j=0}^{\infty}$ be the sequence of functions with resolutions $j = 0, 1, \cdots$, further letting the centers be strictly in ascending orders and by $\sum_{k\in\mathbb{Z}} \psi_{j,k} = 1$ for $j = 0, 1, \cdots$, we have that

$$g_j(x) = \sum_{k\in\mathbb{Z}} f(x_{j,k})\psi_{j,k}(x) \quad for\ x \in \mathbb{R},$$

where $\psi_{j,k}(x)$ is basis functions defined above.

$$|f(x) - g_j(x)| = |f(x) - \sum_{k\in\mathbb{Z}} f(x_{j,k})\psi_{j,k}(x)|.$$

Consequently, we have

$$|f(x) - g_j(x)| \leq \sum_{k\in\mathbb{Z}} |f(x) - f(x_{j,k})|\psi_{j,k}(x).$$

In order to bound the difference of f, we let $\{\omega(f, 2^{-j}) : j \to \infty\}$ be the modulus of continuity of f. Therefore, we have

$$|f(x) - g_j(x)| \leq C\omega(f, 2^{-j}),$$

where C is some positive constant independent of j and dependent of f. Then by Corollary 4.3.1 the result follows. ∎

Hence, a necessary condition for the corresponding quasi–interpolation scheme uniformly convergent is that the $\max_k |d_{j,k}|$ is of order of $2^{-j\beta}$, i.e., the approximation property of the function is determined by its smoothness.

We are interested in the smoothness properties of ϕ which determine the smoothness of the components of interpolating function. A scheme is termed C^∞ if $\phi \in C^\infty$, and it is termed H^γ with $0 < \gamma < 1$, if $\phi \in H^\gamma$, namely if

$$|\phi(x) - \phi(y)| \leq C|x - y|^\gamma. \tag{4.20}$$

Thus, the analysis of the convergence of the scheme and the properties of ϕ requires the study the scalar case when dimension of the Euclidean space is one.

Infinite sum does not make any senses with practical applications, hence by Theorem 1 in (Beatson and Powell, 1992) we have the following corollary.

Corollary 4.3.2 *Let $f \in C(\mathbb{R})$ then quasi–interpolating wavelet expansion holds, in the sense of uniform convergence:*

$$\|f - \sum_{j \leq J} \sum_{k \leq K} d_{j,k} \psi_{j,k}\|_\infty \to 0$$

as $J, K \to \infty$.

We thus have a non–orthogonal wavelet decomposition which exhibits explicitly as a measure of error in V_j, for $j \geq j_0$.

4.4 Convergence of Interpolating Wavelet Transform

The purpose of this section is to study the convergence properties of normal multiresolution expansions, and in particular, nonorthogonal wavelet expansions. Unlike Fourier series, a normal wavelet expansion has a summation kernel which is absolutely bounded by dilation of radially decreasing L_1 convulsion kernel $H(|x - y|)$. This fact provide us the proof of L_p, for $1 \leq p \leq \infty$, convergence. This results hold in all dimensions, and apply to related multiscale expansions. The following definition is essential in the sense of boundedness of the projection operator $P_j f$.

Definition 4.4.1 *A function $f(x)$ is in the class $\mathcal{RB}$ if it is absolutely bounded by an L_1 radial decreasing function $\eta(x)$, i.e., $\eta(x_1) = \eta(x_2)$ whenever $|x_1| = |x_2|$, and $\eta(x_1) \leq \eta(x_2)$ when $|x_1| \leq |x_2|$, and $\eta(x) \in L_1(\mathbb{R}^d)$.*

Define the space $\mathcal{F}(\mathbb{R})$ as the space of all functions $\phi \in C^\infty(\mathbb{R})$ such that

$$\upsilon_m(\phi) = \sup_{x \in \mathbb{R}, \alpha < m} (1 + |x|)^N |\phi^\alpha(x)| < \infty, \quad m = 0, 1, \cdots, \quad N > 1,$$

where $D^\alpha = \frac{d^\alpha}{dx^\alpha}$. The topology in $\mathcal{F}(\mathbb{R})$ is defined by the family of the semi–norms υ_m. Then, $\mathcal{F}(\mathbb{R})$ becomes a Fréchet space and the embeddings $\mathcal{D} \hookrightarrow \mathcal{F} \hookrightarrow \mathcal{S} \hookrightarrow \varepsilon$ are continuous; here ε denotes the space of all C^∞ functions, $\mathcal{S}$ the space of the tempered distributions of polynomial growth and $\mathcal{D}$ the space of C^∞ functions with compact supports. By $\mathcal{F}'(\mathbb{R})$, it means that the space of continuous linear functionals on $\mathcal{F}(\mathbb{R})$. A distribution $T \in \mathcal{D}'$ is in $\mathcal{F}(\mathbb{R})$ if and only if there exist positive integers α, m_0 and a bounded continuous function $f(x)$ on $\mathbb{R}$ such that

$$T = D^\alpha[(1+|x|)^N f(x)], \quad N > 1.$$

Now we can define

$$S(\mathbb{R}) = \left\{\theta(t) \in C^\infty(\mathbb{R}) : |D^m\theta(t)| \leq C(1+|t|)^{-N}\right\}, \quad N > 1, \quad m = 0, 1, 2, \cdots,$$

and its dual $S'(\mathbb{R})$.

Let $\phi \in \mathcal{F}(\mathbb{R})$. In order for it to qualify as a scaling function, ϕ must be associated with a multiresolution analysis of $\mathcal{F}$, i.e., a nested sequence of closed subspaces $\{S_j\}_{j\in\mathbb{Z}}$, such that

$$\{\phi(\cdot - k)\} \textit{ is a basis in } S_0,$$

Then, ϕ has the nested property such that

$$\phi_{j,k} = \sum_{m\in I_{j+1}} \phi_{j,k}(m)\phi_{j+1,k}.$$

In this case, there is no mother wavelet $\psi \in \mathcal{F}$. Hence, we restrict our attention into $\phi \in \mathcal{F}(\mathbb{R})$ compactly supported interpolating wavelets of multiquadric of the above.

Remark 4.4.1 *ϕ converges in a compact set $\partial\Omega$ which define a topology. There is only one such topology; it is called the topology of pointwise convergence. Then, Ω is a compact topological vector space; this follows from the Tychonoff theorem.*

The orthogonal wavelet expansion of a function in $L_2(\mathbb{R})$ is known to converge to it in the sense of $L_2(\mathbb{R})$; but the pointwise convergence is a little bit more subtle, Meyer (1992) was among the first to study the convergence of orthogonal wavelet expansions. He showed that if the mother wavelet is r–regular, the orthogonal wavelet expansion of a function will converge to it in the sense of $L_p(\mathbb{R})$; $1 \le p < \infty$; and in the sense of some Sobolev spaces as well.

A function $f(x)$; $x \in \mathbb{R}^d$; $d \ge 1$, is said to be r–regular (in the sense of Meyer) if

$$|D^\alpha f(x)| \le \frac{C_{\alpha,m}}{(1+|x|)^m},$$

for all α with $|\alpha| \le r$ and $m = 0, 1, 2, \cdots$, where $C_{\alpha,m}$ are constants. Here, $\alpha = (\alpha_1, \cdots, \alpha_d)$ is a multi–index with $\alpha_i (i = 1, \cdots, d)$ being a non–negative integer and $|\alpha| = \sum_{i=1}^m \alpha_i$, and

$$D^\alpha = \frac{\partial^\alpha}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}.$$

Assuming that the scaling function ϕ of the multiresolution analysis is r–regular, Walter (1995) proved that the orthogonal wavelet expansion of a function $f \in L_1 \cap L_2$ converges to f pointwise at every point of continuity of f and uniformly on compact subsets of any interval (a, b) on which f is continuous. Later, he relaxed this condition and assumed that ϕ satisfies the condition

$$|\phi(x)| \le \frac{C}{(1+|x|)^3}.$$

Kelly et al. (1994a) improved Walter's results by proving pointwise convergence of orthogonal wavelet expansions not only under less stringent conditions, but also by extending them to n–dimensions. Crucial to their proofs is the following definition.

They showed, among other things, that if ϕ, ψ^λ are in the class $\mathcal{RB}$ for all λ, then the wavelet of any function $f \in L_p(\mathbb{R}^d)$ converge to f pointwise almost everywhere. Moreover, if only $\psi^\lambda \ln(2 + |x|) \in \mathcal{RB}$, then the wavelet expansion of any function $f \in L_p(\mathbb{R}^d)$ converges to f pointwise almost everywhere. In all the work cited above on pointwise convergence; it is essential that the summation kernel of the wavelet series

$$P_m(x, y) = \sum_{j<m;k,\lambda} \psi^\lambda_{m,k}(x)\overline{\psi^\lambda_{m,k}}(y) = \sum_k \phi_{m,k}(x)\overline{\phi_{m,k}}(y)$$

be absolutely bounded by a dilation of an L_1–radially decreasing convolution kernel, i.e., by $H(2^m|x - y|)$, where $H \in \mathcal{RB}$.

Following Walter, ϕ is regular if there exist a $c > 0$ such that $|\phi(x)|$, $|\phi(x)| \le c/(1+|x|)^3$ for all $x \in \mathbb{R}$. Note that for any regular wavelet ϕ, we have $\phi \in \mathcal{RB}$. Here we consider larger classes of scaling functions with polynomial order of decay $N > 1$. We have the following regularity of scaling function

$$\phi(x) = C(1 + |x|)^{-1-\varepsilon}, \quad \varepsilon > 0,$$

for some suitably chosen C (all of which include the above class).

We shall assume that the scaling function and the wavelet of the normal multiresolution analysis, ϕ is in L_1, and we shall not assume that they generate an orthonormal basis of V_0. This means that unlike the wavelet expansions studied in (Kelly et al., 1994a,b; Walter, 1995), ours are not necessarily orthonormal (Zayed, 2000). Therefore, it is replaced by the weaker condition that $\{\phi(x-k)\}$ is an unconditional Schauder basis of V_0 and it is not assumed that ϕ generates an orthonormal basis, but on the other hand it is assumed a stronger condition on ϕ, namely, that its Fourier transform $\hat{\phi}$ has a compact support.

Lemma 4.4.1 *Let $\phi_{j,k}$ be the scaling function in V_j then there exist a summation kernel such that*

$$P_j(x, y) = \sum_{k \in 2^{-j}\mathbb{Z}} \phi_{j,k}(x)\phi^{\star}_{j,k}(y), \quad for\ all\ x, y \in \mathbb{R}. \tag{4.21}$$

Proof : Since $\{\phi(x-k)\}$ is an unconditional Schauder basis of V_0, it has a biorthonormal basis $\{\phi^{\star}_k(x)\}$ such that for any $f \in V_0$ we have

$$f(x) = \sum_k \langle f, \phi^{\star}_k \rangle \, \phi_{0,k}(x),$$

in the sense of $L_2(\mathbb{R})$, where $\phi_{0,k} = \phi(x-k)$. Similarly, for any $f \in V_m$ and for fixed m, we have

$$f(x) = \sum_k \left\langle f, \phi^{\star}_{m,k} \right\rangle \phi_{m,k}(x),$$

where $\{\phi^{\star}_{m,k}(x)\}_{k=-\infty}^{\infty}$ is the biorthonormal basis of $\{\phi_{m,k}(x)\}_{k=-\infty}^{\infty}$. From the relation

$$\langle \phi_m, \phi^{\star}_k \rangle = \delta_{k,m} = \int_{\infty}^{\infty} \phi(x-k)\overline{\phi^{\star}_k}(x)dx,$$

we obtain by a change of variable that

$$\langle \phi_m, \phi^{\star}_k \rangle = \delta_{k,m} = 2^m \int_{\infty}^{\infty} \phi(2^m x - k)\overline{\phi^{\star}_k}(2^m x)dx.$$

It follows that $\{2^{m/2}\phi^{\star}_{m,k}(2^m x)\}_{k=-\infty}^{\infty}$ is the biorthonormal basis of $\{\phi_{m,k}(x)\}_{k=-\infty}^{\infty}$ for each m. Hence the result. ∎

Since we assume that $\{\phi(x-k)\}$ is an unconditional Schauder basis of V_0 and Fourier transform $\hat{\phi}$ has a compact support, for such classes, the summation kernel satisfy the following bounds.

Theorem 4.4.1 *Let $P_j(x,y) = \sum_{k\in 2^{-j}\mathbb{Z}} \phi_{j,k}(x)\phi^\star_{j,k}(y)$ be the summation kernel generated by the scaling function $\phi \in L_1(\mathbb{R})$. If ϕ has an algebraic decay such that $\phi(x) \le \frac{C_N}{(1+|x|)^N}$ for some $N > 1$ then*

$$|P_j(x,y)| \le C_N \frac{2^j}{(1+2^j|x-y|)^N} \le C_N 2^j,$$

for some $N > 1$.

Proof : Similar lines of proof of Theorem 2.5 as in Kelly et al. (1994a).

The space V_0 is a reproducing–kernel Hilbert space since point evaluation is continuous.

$$|f(x)| = |\frac{1}{\sqrt{2\pi}} \int_E \hat{f}(\omega)e^{-i\omega x}d\omega| \le C\|\hat{f}\| = C\|f\|.$$

Therefore, it has a reproducing kernel $k(t,x)$, which is easily seen to be a convolution kernel. For, if $f \in V_0$, then

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_E \hat{f}(\omega)e^{-i\omega x}d\omega| = \int_{-\infty}^{\infty} f(t)k(t,x)dt, \tag{4.22}$$

where

$$k(t,x) = \frac{1}{2\pi}\int_E e^{i\omega(t-x)}d\omega = k(t-x).$$

The last integral in (4.22) is absolutely convergent by the Cauchy–Schwartz inequality because both f and k are in $L_2(\mathbb{R})$. The reproducing kernel has also the representation $k(t,x) = \sum_n \phi^\star_n \phi(x-n)$, where the series converges in the sense of L_2. But from the hypothesis, the series also converges absolutely and uniformly for all x and t to a function $q_0(t,x)$. Thus, by standard arguments, $q_0(t,x) = k(t,x)$ converges almost everywhere. One can also show directly that if $f \in V_0$, then

$$\int_{-\infty}^{\infty} f(t)k(t,x)dt = \sum_n \left(\int_{-\infty}^{\infty} f(t)\phi^\star_n dt\right)\phi(x-n)$$

$$= \sum_n (f, \phi^\star_n)\phi(x-n) = f(x).$$

Since strong convergence implies weak convergence and, in addition, strong convergence in a reproducing–kernel Hilbert space implies pointwise convergence, as well as uniform convergence on any set on which $k(x,x)$ is uniformly bounded.

The reproducing kernel of V_0 is given by

$$P_0(x,y) = \sum_k \phi(x-k)\overline{\phi(y-k)},$$

where ϕ is the scaling function. The series and its derivatives with respect to k converge uniformly on $x \in \mathbb{R}$ because of the regularity of $\phi \in \mathcal{F}$, i.e.,

$$|\phi^\alpha(x)| \leq C(1+|x|)^{-N}, \quad \alpha = 0,1,2,\cdots, \quad N > 1.$$

Then, associated with the increasing sequence of subspaces $\{V_j\}_{j\in\mathbb{Z}}$ we have the projections of $L_2(\mathbb{R})$ onto V_j given by

$$P_j f = \sum_k \left\langle f, \phi^\star_{j,k} \right\rangle \phi_{j,k}, \quad for\ f \in L_2(\mathbb{R}).$$

From now on, we shall work with a scaling function ϕ in the class of $\mathcal{RB}$; that means that ϕ is absolutely bounded by an L_1 radial decreasing function η, i.e., $|\phi(x)| \leq \eta(x)$ with $\eta(x)(0) < \infty$, $\eta(x1) = \eta(x2)$ whenever $|x1| = |x2|$, $\eta(x1) < \eta(x2)$ whenever $|x1| > |x2|$ and $\eta \in L_1(\mathbb{R})$. Under the assumption $\phi \in \mathcal{RB}$ we get that the kernel $P_j(x,y)$ of P_j is given by $2^j P_0(2^j x, 2^j y)$ with $P_0(x,y) = \sum_k \phi(x-k)\overline{\phi(y-k)}$, in the sense that, for $f \in L_2(\mathbb{R})$.

$$P_j f(x) = \int_{\mathbb{R}} P_j(x,y) f(y) dy. \tag{4.23}$$

In (Kelly et al., 1994a) proved that if $\phi \in \mathcal{RB}$ then the kernel $P_0(x,y)$ satisfies

$$|P_0(x,y)| \leq H(x-y), \tag{4.24}$$

where $H(|x|)$ is a bounded radial decreasing $L_1(\mathbb{R})$ function (see Lemma 2.8 in Kelly et al. (1994b)). From the estimate (4.23), (Theorem 2.6 [11] in Kelly et al. (1994b)), the operators $P_j f(x) = \int_{\mathbb{R}} P_j(x,y) f(y) dy$ are well defined for $f \in L_p(\mathbb{R})$, $1 \leq p < \infty$, and $P_j f$ converge to f almost everywhere and in the L_p norm, when $j \to \infty$.

Note that $\overline{\phi}$ is well defined in the linear space S_0 and supported in the a compact set Ω. Moreover, these functions belong to L_p, for $1 \leq p \leq \infty$. Then the kernel $P_m(x,y)$ has the form

$$P_m(x,y) = \sum_k \phi_m(x-k)\overline{\phi_m(y-k)}, \tag{4.25}$$

i.e., for every $x, y \in \mathbb{R}^d$, with convergence of both sums in the right occurring pointwise on compact subsets of a positive distance from the diagonal $D = \{(x,y) : x = y\}$. The kernel converges to a delta distribution $\delta(x-y)$ as $m \to \infty$ in the following sense :

Theorem 4.4.2 *Under the assumption that $\phi_{j,k} \in \mathcal{RB}$, the kernel $P_j(x,y)$ of the projections onto S_j satisfy the convulsion bound*

$$|P_j(x,y)| \leq C\ 2^j\ H(2^j|x-y|), \tag{4.26}$$

where $H(|\cdot|) \in \mathcal{RB}$, i.e., $H(|\cdot|)$ is a radially decreasing L_1 function.

Proof of this theorem directly follows form Theorem 2.6 [11] in Kelly et al. (1994b).

The sum on the right converges absolutely under our assumption. The integral also absolutely convergent for $f \in L_p(\mathbb{R})$, $1 \leq p \leq, \infty$ which will follow from the fact that $P_m(x,y)$ is bounded by a convulsion kernel H. For $f \in L_1(\Omega)$, x in the Lebesgue set of f, the wavelet expansion of f to $f(x)$ at x, i.e.,

$$\lim_{m\to\infty} P_m f(x) = \sum_{m\to\infty}\sum_k c_{j,k}\phi_{j,k}(x) = f(x),$$

where $c_{j,k}$ is the wavelet coefficients, in particular, which is determined by the the Lebesgue set of f. Furthermore, if f is uniformly continuous, then the convergence is uniform.

4.5 Summary

It is shown that there is compactly supported function for normal multiresolution approximation such that the scaling function and wavelet function (based on the hypothesis of interpolating wavelet basis function) on real line. Clearly, the basis function defines a triangular function. Moreover, pointwise convergent property of the function f in $L_p(\mathbb{R})$, $1 \leq p \leq \infty$ is proved.

Hence, with the assumption of uniform cone condition of a domain $\Omega \subset \mathbb{R}^d$, $d \geq 1$, function decomposition into normal wavelet is developed in the next chapter. The principle component of this scheme is that there is a 1–unisolvent set $X_j = \{x_1^j, \cdots, x_n^j\}$ for each resolution level j, as the data sets are on the boundary $\partial\Omega$ of the domain Ω.

Chapter 5

Normal Wavelets in a Bounded Domain

5.1 Interpolating Functions

The theory of wavelets and multiresolution analysis is usually developed on $\mathbb{R}^d$ while applications of wavelets to image processing and numerical methods for partial differential equations require multiresolution analysis on domains or manifolds on $\mathbb{R}^d$. Many real life problems require algorithms adapted to irregular sampled data while first generation wavelets imply a regular sampling of the data. Moreover, Diagonalization of differential forms, analysis on curves and surfaces, and weighted approximation require a basis adapted to weighted measures. However, first generation wavelets typically provide bases only for spaces with translation invariant (Haar–Lebesgue) measures defined on $\mathbb{R}^d$. Hence, a proper substitute is needed.

A generalization of first generation wavelets while preserving the their properties is known as second generation wavelets. In this setting our concern is normal multiresolution approximation and its basis functions. In the previous chapter we have shown that multiquadric (in odd dimensional Euclidean apace) functions may be an ideal choice for interpolating wavelet basis functions with polynomial reproduction property. The normal multiresolution approximation of a function f (with some smoothness properties) have interpolating basis functions, known as multiquadric and inverse multiquadric functions. That is, former property could be generalized to later. This is the core of our approach to find basis functions for normal multiresolution approximation.

5.2 Construction of Normal Subdivision Scheme

Let Ω be a compact subset of $\mathbb{R}^d$, and let $f \in C(\Omega)$ be a function decomposed by normal multiresolution approximation. This scheme produces one irregular point set as opposed to the traditional wavelet transform. In the irregular case, the subdivision scheme becomes both spatially variant and non stationary. Smoothness results are not straightforward; because the subdivision is spatially variant. The Fourier transform can no longer be used because it is non stationary, even spectral analysis cannot help.

Much of this effort hinges on the idea of multiresolution analysis as a device to construct wavelet approximation likewise traditional wavelet transform. In this setup, there is a nested sequence of spaces

$$\mathcal{S}_0 \subset \mathcal{S}_1 \subset \mathcal{S}_2 \subset \cdots,$$

from which a given function is approximated. The hypothesis of this work is that the wavelet is simply a finer scale scaling function at odd–locations.

Normal multiresolution induces a parameterizations of a curve f as depicted in Figure 2.3 in Chapter 2. Analytically this parameterizations is as follows: we define at every level j, $s_j : [0,1] \to \mathbb{R}^d$ to be piecewise affine map with breakpoints at the $t_{j,k} = 2^{-j}k$, $k = 0, 1, \cdots, 2^j$ and for which $f(s_j(t_{j,k})) = v_{j,k}$, i.e., $s_j(t_{j,k}) = s_{j,k}$.

Definition 5.2.1 *If the node points on $f_{j,k}$ approach each other when $j \to \infty$ then $s_{j,k}(t)$ converges absolutely to a function $s(t)$, i.e., the limit function,*

$$\lim_{j\to\infty} \sup_k |f_{j,k} - s_{j,k}| = 0.$$

The parameterization of a function f induced by the normal multiresolution maps $t \in [0,1]$ to $f(s(t))$; we shall call this the normal parameterization of a function f. The regularity of normal parameterizations is related to the decay of wavelet coefficients which directly determines the approximation quality. It is easy to see the above scheme defines a polygonal boundary $\partial\Omega$ in $\Omega \subseteq \mathbb{R}^d$ and polygon can be decomposed as triangles at subsequent resolution levels. Hence, let $\triangle_{\mathcal{NS}}$ be a normal subdivision triangulation of a subset $\Omega \subseteq \mathbb{R}^d$ with polygonal boundary $\partial\Omega$.

5.2.1 Polygonal Domain

In this section we introduce some notation that will be needed later. Then we review some basis properties of triangulation.

For a positive integer n, $\mathbb{R}^n$ denote the n–dimensional Euclidean space with inner product given by

$$x.y := x_1y_1 + \cdots + x_ny_n \quad for\ x = (x_1, \cdots, x_n)\ and \quad y = (y_1, \cdots, y_n) \in \mathbb{R}^n.$$

Consequently, the norm of a vector $x \in \mathbb{R}^n$ is given by $|x| := (x.x)^{1/2}$.

An element of $\mathbb{N}_0^n$ is called multi–index. The length of a multi–index $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}_0^n$ is given by $|\alpha| := \alpha_1 + \cdots + \alpha_n$. For $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}_0^n$ and $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, define

$$x^\alpha := (x_1^{\alpha_1}, \cdots, x_n^{\alpha_n}) \in \mathbb{R}^n.$$

The function $x \mapsto x^\alpha$, $(x \in \mathbb{R}^n)$, is called a monomial and its (total) degree is $|\alpha|$. A polynomial is a linear combination of monomials. The degree of a polynomial q is defined to be deg $q := \max\{|\alpha| : c_\alpha \neq 0\}$. For an integer $k \geq 0$, we use Π_k to denote linear space of all polynomials of degree at most k.

For vectors $y = (y_1, \cdots, y_n) \in \mathbb{R}^n$, we use D_y to denote the differential operator given by

$$D_y f(x) := \lim_{t \to 0} \frac{f(x + ty) - f(x)}{t}, \quad x \in \mathbb{R}^n.$$

Let $e_1, \cdots, e_n$ be the unit coordinate vectors in $\mathbb{R}^n$. For $j = 1, \cdots n$, we write D_j for D_{x_j}. For multi–index $\alpha = (\alpha_1, \cdots, \alpha_n)$, D^α stands for differential operator $D_1^{\alpha_1} \cdots D_n^{\alpha_n}$.

Maximal operators play an important role in interpolation and differentiation. A paradigm is that so called sharp maximal functions of Calderón and Scott, given by

$$f_\alpha^\sharp := \sup_{Q \ni x} \frac{1}{|Q|^{1+\alpha/d}} \int_Q |f - f_Q|, \quad 0 < \alpha < 1, \tag{5.1}$$

where $f_Q := |Q|^{-1} \int_Q f$ is the average of f over the cube Q, and Q ranges over all x. When $\alpha > 0$, $f_\alpha^\sharp$ is related to classical differentiation; for instance it is well known that

$$f \in Lip_\alpha(\mathbb{R}^n) \Leftrightarrow f_\alpha^\sharp \in L_\infty(\mathbb{R}^n), \quad 0 < \alpha < 1,$$

where Lip_α is the Lipschitz space of smoothness α.

The extension of (5.1) to functions of higher smoothness $\alpha \geq 1$ is given by replacing the average f_Q by the best polynomial approximation from Π_α, the space of polynomial degree at most of α. Now, let Ω be a (Lebesgue) measurable subset of $\mathbb{R}^n$. Suppose f is a (real valued) measurable function on Ω. For $1 \leq p < \infty$, let

$$\|f\|_{p,\Omega} := \left(\int_\Omega |f(x)|^p dx \right)^{1/p}.$$

For $p = \infty$, let $\|f\|_{\infty,\Omega}$be the essential supremum of $|f|$ on Ω. By $L_p(\Omega)$ $(1 \leq p \leq \infty)$ we denote the linear space of all functions f on Ω such that $\|f\|_{p,\Omega} < \infty$. Equipped with the norm $\|\cdot\|_{p,\Omega}$, $L_p(\Omega)$ becomes Banach space.

Let $\mathcal{T}$ be a finite collection of triangles in $\mathbb{R}^n$. The intersection of any two triangles in $\mathcal{T}$ is empty, or a common vertex, or a common edge of them. Let Ω be the union of the triangles in $\mathcal{T}$. Then, Ω is a polygonal domain in $\mathbb{R}^n$ and $\mathcal{T}$ is a triangulation of Ω. The domain Ω could be bounded or unbounded. But, Ω is always assumed to be a Lipschitz–graph domain.

5.3 Interpolating Basis Functions

Radial basis functions are well–known and useful tool for functional approximation in one or more dimensions. The general form of approximation is always a linear combination (finite or infinite) of number of shifts of a single function, the radial basis function. In more than one dimension, this function is made rotationally invariant by composing a univariate function, usually called φ, with the Euclidean norm. In one dimension such approximation usually simplifies to univariate polynomial splines.

This section present compactly supported quasi–interpolating basis function which interpolates the given equally spaced centers. This function is scale invariant, i.e., the basis function is the generating function of the scale invariant space. Moreover, the functions is scale invariant, i.e., a single function is used over different resolution level.

In as much as multiquadric functions interpolations are used mainly for the purpose of multiresolution approximation in applications, the assumption that the data lie at the vertices of an infinite regular grid gives rise to an interesting special case. The $L_p(\mathbb{R}^d)$, $d > 0$, approximation order is at least μ for approximants from an h–dependent space $\mathcal{S} = \mathcal{S}_h$ of the approximants with centers $h\mathbb{Z}^d$, if

$$dist_{L_p(\mathbb{R}^d)}(f, \mathcal{S}) := \inf_{g \in \mathcal{S}} \|f - g\|_p = O(h^\mu), \quad for\ all\ f,$$

from a given space of approximants. The h–dependent of $\mathcal{S}$ comes only from the fact the shifts of the radial basis function on scaled integer grid $h\mathbb{Z}^d$.

The interpolants are defined on equally spaced grids as;

$$\mathbb{Z}^d = \{(k_1, k_2, \cdots, k_d) : k_i \in \mathbb{Z}, i = 1, 2, \cdots, d\},$$

or $h\mathbb{Z}^d$ is that they are periodic and boundary–free. The space spanned by shifts of basis function, call it ψ, namely by

$$\psi(\cdot - k), \quad k \in \mathbb{Z}^d,\ d > 0,$$

are called shift–invariant because of any f in such space, it shifts $f(\cdot - k)$, $k \in \mathbb{Z}^d$ is an element of the space.

The goal of interpolations on grids is to find interpolants which in our case; the construction admits in particular all pointwise absolutely convergent sums (with real coefficients d_k), and more precisely, everywhere on the Lebesgue set of function being expanded, have the form

$$\sum_{k \in \mathbb{Z}^d} d_k \psi(x - k), \quad x \in \mathbb{R}^d,$$

where $f : C(\mathbb{R}^d) \to \mathbb{R}$ is the function we wish to interpolate and ψ has the following expansion

$$\psi(x) = \sum_{\mu \in \mathbb{Z}^d} c_\mu \varphi(\|x - \mu\|), \quad x \in \mathbb{R}^d,$$

which is finite.

All sums are being assumed at present to be absolutely convergent in a compact set K, φ being multiquadric and $\{c_\mu\}_{\mu\in\mathbb{Z}^d}$ be suitable f–independent that is of compact support with respect to μ. So far, we have written an interpolant based on $\mathbb{Z}^d$ translate of $\varphi(\|\cdot\|)$ but to get convergence on the whole underlying space of $\mathbb{R}^d$ of the interpolant f, it is necessary to base the interpolant on $h\mathbb{Z}^d$, h being positive and becoming small. In fact, it is desirable in the latter case remain with exactly same ψ, but then we must scale the argument of the ψ as follows:

$$\sum_{k\in\mathbb{Z}^d} hd_{j,k}\psi(x/h-k), \quad x\in\mathbb{R}^d,$$

where $hd_{j,k}$ is scaled coefficient sequence at the resolution level $j\in h\mathbb{Z}^d$. In the language of shift–invariant spaces and multiresolution analysis is a stationary scaling, since for all h the same function ψ is used which is scaled by h^{-1} inside its argument.

5.3.1 Compactly Supported Scaling Function

In this section, we wish to develop more general approach employing the concepts of traditional wavelets and radial basis functions and employ shift–invariant spaces of approximation for our interpolating wavelet transforms. To start with, we wish to find a function $\phi\in L_1(\mathbb{R})$ which satisfies the following decay condition

$$|\hat{\phi}(\xi)|\le C(1+|\xi|)^{-3-\varepsilon},$$

and the partition of unity conditions, where C is positive generic constant and $\varepsilon>0$. $\hat{\phi}$ denotes the Fourier transform. Since, we demand the interpolating basis function must have the property that $\hat{\phi}(0)\neq 0$. Fourier transform $\hat{\phi}$ is continuous and well defined. Without lose of generality we assume that $\hat{\phi}(0)=1$. Hence, the interpolating function is well defined and in a shift–invariant space.

The function $f\in C(\Omega)$ can be interpolated with multiquadric function $\varphi(r)=\sqrt{r^2+c^2}$, where $r=\|x\|$ is the Euclidean norm and c is a constant. Define

$$\phi(t)=\frac{1}{2}\varphi''(t), \quad \textit{for all } t\in\mathbb{R}^d. \tag{5.2}$$

The function $\phi \in C_0^\infty$ is the quasi–interpolating basis function which interpolates the function f^j associated with the compact set Ω_j which satisfy the properties of multiresolution defined below in Section 5.5. The interpolating sequence $\{\phi_k : k \geq 0\} \subset C_0^\infty$ is now defined as

$$\phi_k = \delta_{0,k}, \quad for\ all\ k \in \mathbb{Z}^d.$$

Then we assume that the function is in the generalized smoothness of function spaces. Hence, the following section describe the sequence of functions admissible in the sense of generalized smooth function spaces.

5.3.2 Admissible Sequence of Functions

We shall adopt the following general notation: $\mathbb{N}$ denotes the set of all natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}^n$, $n \in \mathbb{N}$, denotes the n–dimensional real Euclidean space and $\mathbb{R} = \mathbb{R}^1$. We use the equivalence $\sim$ in

$$a_k \sim b_k \quad or \quad \varphi(r) \sim \psi(r),$$

always to mean that there are two positive numbers c_1 and c_2 such that

$$c_1 a_k \leq c_2 b_k \quad or \quad c_1 \varphi(x) \leq \psi(x) \leq c_2 \varphi(x),$$

for all admitted values of the discrete variable k or the continuous variable x, where $(a_k)_k$, $(b_k)_k$ are non–negative sequences and φ, ψ are non–negative functions. If $a \in \mathbb{R}$ then $a_+ := \max(a, 0)$. Here we consider n–dimensional Euclidean space and real line interchangeably then we use the notation $\mathbb{R}$ invariably. We explain the class of sequences we shall be interested in and some related basic results.

A sequence $\gamma = (\gamma_j)_{j \in \mathbb{N}_0}$ of positive real numbers is said to be admissible if there exist two positive constants d_0 and d_1 such that

$$d_0 \gamma_j \leq \gamma_{j+1} \leq d_1 \gamma_j, \quad j \in \mathbb{N}_0. \tag{5.3}$$

For an admissible sequence $\gamma = (\gamma_j)_{j \in \mathbb{N}_0}$, let

$$\underline{\gamma}_j := \inf_{k \geq 0} \frac{\gamma_{j+k}}{\gamma_k} \quad and \quad \overline{\gamma}_j := \sup_{k \geq 0} \frac{\gamma_{j+k}}{\gamma_k}, \quad j \in \mathbb{N}_0. \tag{5.4}$$

Clearly, $\underline{\gamma}_j\gamma_k \leq \gamma_{j+k} \leq \overline{\gamma}_j\gamma_k$ for any $j,k \in \mathbb{N}_0$. In particular, $\underline{\gamma}_1$ and $\overline{\gamma}_1$ are the best possible constants d_0 and d_1 in (5.3) respectively. Then the lower and upper Boyd indices of the sequence γ are defined respectively by

$$\underline{s}(\gamma) := \lim_{j\to\infty} \frac{\log \underline{\gamma}_j}{j} \quad \textit{and} \quad \overline{s}(\gamma) := \lim_{j\to\infty} \frac{\log \overline{\gamma}_j}{j}. \tag{5.5}$$

The above definition is well posed: the sequence $(\log \overline{\gamma}_j)_{j\in\mathbb{N}}$ is sub–additive and hence the right–hand side limit in (5.5) exists, it is finite (since γ is an admissible sequence) and it coincides with $\inf_{j>0} \log \overline{\gamma}_j/j$. The corresponding assertions for the lower counterpart $\underline{s}(\gamma)$ can be read off observing that $\log \underline{\gamma}_j = -\log(\overline{\gamma_j^{-1}})$.

The Boyd index $\overline{s}(\gamma)$ of an admissible sequence γ describes the asymptotic behavior of the $\overline{\gamma}_j$'s and provides more information than simply $\overline{\gamma}_1$ and, what is more, is stable under the equivalence of sequences: if $\gamma \sim \tau$, then $\overline{s}(\gamma) = \overline{s}(\tau)$ as one readily verifies. The same applies to the lower counterpart. Observe also that for each $\epsilon > 0$ there are two positive constants $c_1 = c_1(\epsilon)$ and $c_2 = c_2(\epsilon)$ such that

$$c_1 2^{(\underline{s}(\gamma)-\epsilon)j} \leq \underline{\gamma}_j \leq \overline{\gamma}_j \leq c_2 2^{(\overline{s}(\gamma)-\epsilon)j}, \quad j \in \mathbb{N}_0. \tag{5.6}$$

Let $N = (N_j)_{j\in\mathbb{N}_0}$ be an admissible sequence with $\underline{N}_1 > 1$ (recall (5.4)). In particular N is a so–called strongly increasing sequence which guarantees the existence of a number $k_0 \in \mathbb{N}_0$ such that

$$N_k \geq 2N_j, \quad \textit{for any } k,j \textit{ such that} \quad k \geq j + k_0. \tag{5.7}$$

It should be noted that the sequence $N = (N_j)_{j\in\mathbb{N}_0}$ plays the same role as the sequence $(2^j)_{j\in\mathbb{N}_0}$ in the classical construction of function spaces, such as Besov spaces, $B^s_{p,q}$.

For a fixed sequence $N = (N_j)_{j\in\mathbb{N}_0}$, as above, we define the associated covering $\Omega^N_j = (\Omega^N_j)_{j\in\mathbb{N}_0}$ of $\mathbb{R}$ by

$$\Omega^N_j = \{x \in \mathbb{R} : |x| \leq N_{j+k_0}\}, \quad j = 0, \cdots, k_0 - 1,$$

and

$$\Omega^N_j = \{x \in \mathbb{R} : N_{j-k_0} \leq |x| \leq N_{j+k_0}\}, \quad j \geq k_0,$$

with k_0 according to (5.7).

Definition 5.3.1 *For a fixed increasing sequence $N = (N_j)_{j\in\mathbb{N}_0}$ with $\underline{N}_1 > 1$ and for the associated compact set $\Omega^N = (\Omega_j^N)_{j\in\mathbb{N}_0}$ of $\mathbb{R}$, a system $\phi^N = (\phi_j^N)_{j\in\mathbb{N}_0}$ will be called a (generalized) partition of unity subordinated to Ω^N if :*

1. *$\phi_j^N \in C_0^\infty$ and $\phi_j^N(x) \geq 0$ if $x \in \mathbb{R}$ for any $j \in \mathbb{N}_0$;*
2. *supp $\{\phi_j^N\} \subset \Omega_j^N$ for any $j \in \mathbb{N}_0$;*
3. *for any $\alpha \in \mathbb{N}_0^n$ there exist a constant c_α such that for any $j \in \mathbb{N}_0$*
$$|D^\alpha \phi(x)| \leq c_\alpha (1 + |x|)^{-R}, \quad R \geq 1, \quad for\ any\ x \in \mathbb{R};$$
4. *there exist a constant c_ϕ such that*
$$0 < \sum_{k\in\mathbb{Z}} \phi_j^N(x) = c_\phi < \infty, \quad for\ any\ x \in \mathbb{R}.$$
Without lose of generality we assume that $c_\phi = 1$.

Example We consider example of an admissible sequences. Let $\phi : (0, 1] \to \mathbb{R}$ be a slowly varying function. Then, for each $s \in \mathbb{R}$ the sequence $\gamma = (2^{sj}\phi(2^{-j}))_{j\in\mathbb{N}_0}$ is an admissible sequence. Also, here we have $\underline{s}(\gamma) = \overline{s}(\gamma) = s$.

Example The case $\gamma = (2^{sj}\psi(2^{-j}))_{j\in\mathbb{N}_0}$, where ψ is an admissible function in the sense of Triebel (1999) (i.e., a positive monotone function defined on $(0, 1]$ such that $\psi(2^{-2j}) \sim \psi(2^{-j})$, $j \in \mathbb{N}_0$), can be regarded as a special case of the above example.

Remark 5.3.1 *The examples above have in common the fact that their upper and lower Boyd indices coincide. However, this is not in the general case, i.e., an admissible sequence has not necessarily a fixed main order.*

5.4 Multilevel Triangulation

In the previous section we have shown that a function or a surface can be represented by refinement of $\triangle_{j+1}$ of $\triangle_j$ with some subdivision scheme. The subscript j denotes the resolution level. Subdivision schemes are used as prediction step in construction of second generation wavelets. Usually the second generation wavelets are considered as interpolating wavelet transform. Hence, there is an interpolating function known as scaling function and also the wavelet function under some mild assumption.

These $\triangle$ can be considered as scaling functions and, these scaling functions are constructed with translation of one compactly supported interpolating function as the linear combination of multiquadric function in scale invariant space. This is the case with normal multiresolution scheme. This normal multiresolution subdivision scheme yields nested sequences of

$$\triangle_0^{\mathcal{NS}} \subset \triangle_1^{\mathcal{NS}} \subset \triangle_2^{\mathcal{NS}} \subset \cdots,$$

that are regular in the sense of definition stated above. In this scheme, we assume that the regular parameterization is used then the mesh size is equal at each resolution level.

A set $F \subset \mathbb{R}^d$ is said to be a compact polygonal domain if F can be represented as the union of a finite set $\mathcal{T}_0$ of closed triangles with disjoint interiors: $F = \bigcup_{\triangle \in \mathcal{T}_0} \triangle$. Weak locally regular, locally regular, etc. triangulations $\mathcal{T} = \bigcup_{m \geq 0}^{\infty} \mathcal{T}_m$ of such domain $F \subset \mathbb{R}^d$ are defined similarly as when $F = \mathbb{R}^d$. The only essential distinctions are that the levels $(\mathcal{T}_m)$ now are consecutive refinements of the initial coarse level $\mathcal{T}_0$ and, if a vertex $v \in \mathcal{V}_m$ is on the boundary, we should include in $\mathcal{V}_m$ as many copies of v as is its multiplicity.

Remark 5.4.1 *It is a key observation that the collection of all regular sequence of triangulations with given (fixed) parameters is invariant under affine transforms.*

Affine transform angle condition: There exists a constant $\beta = \beta(T)$, $0 < \beta \leq \pi/3$, such that if $\triangle_0 \in \mathcal{T}_m$, $m \in \mathbb{Z}$ and $A : \mathbb{R}^d \to \mathbb{R}^d$ is an affine transform that maps $\triangle_0$ one–to–one onto an equilateral reference triangle, then for every $\triangle \in \mathcal{T}_m$ which has at least one common vertex with $\triangle_0$, then we have

$$\min \ angle(A(\triangle)) \geq \beta, \tag{5.8}$$

where $A(\triangle)$ is the image of $\triangle$ by the affine transform A. In sequel, the following notation is used: for a given triangulation $\mathcal{T}_m$, let $\sigma_{m,k}$ denotes the set of all k–dimensional subsimplices of simplices in $\mathcal{T}_m$, and put $S_{m,k} = \bigcup\{\triangle : \triangle \in \sigma_{m,k}\}$ and, for a given

$\triangle \in \sigma_{m,k}$,

$$\triangle^* := S^{\triangle}_{m,k} = \bigcup \{\triangle_1 : \triangle_1 \in \sigma_{m,k}, \triangle_1 \neq \triangle\}.$$

Hence the following lemma.

Lemma 5.4.1 *Let $k > 0$ be an integer. Suppose that $\tau \in \triangle^*$ is a triangle in $\mathbb{R}^d$, θ is the minimum of the angle of τ. Then we have*

$$C_1(2^{-j})^{d/p-1}\|f\|_{\infty,\tau} \leq \|f\|_{p,\tau} \leq C_2(2^{-j})^{d/p-1}\|f\|_{\infty,\tau}, \quad for\ all f \in \Pi_k|_\tau \quad 1 \leq p \leq \infty, \tag{5.9}$$

where C_1 and C_2 are two positive constants depending only on k and θ.

Proof : Let T be a triangle or some arbitrary partition in $\mathbb{R}^d$ represented by some polynomial $\pi \in \Pi_{k|T}$, then $\Pi_{k|T}$ is finite dimensional space. Since any two norms on finite dimensional space are equivalent, there exist positive constants A and B such that

$$A\|g\|_{\infty,\tau} \leq \|g\|_{p,\tau} \leq B\|g\|_{\infty,\tau}, \quad for\ all\ g \in \Pi_k|_\tau \quad 1 \leq p \leq \infty.$$

Hence the result. ∎

5.4.1 Triangulation of a Function

The construction of a multiresolution analysis over a triangulation is closely related to the nested spline spaces. Since by remark (4.3.2) in Chapter 4, we know that each function ϕ is a triangular function, then we consider the functions define triangular sequence or the functions represent the triangular sequence.

Definition 5.4.1 *Let F be a compact subset of $\mathbb{R}^d$ defines a finite set $\mathcal{T}$ of d–dimensional (non degenerate) simplices is called a triangulation of F if the following conditions hold.*

A1. For each pair $\triangle_i, \triangle_{i+1} \in \mathcal{T}$, $\triangle_i \neq \triangle_{i+1}$, for some $i \in \mathbb{N}_0$ the intersection $\triangle_i \cap \triangle_{i+1}$ is either empty or a common face of lower dimension.

A2. Every vertex of simplex $\triangle \in \mathcal{T}$ is in F.

A3. $F \subset \bigcup_{\triangle \in \mathcal{T}} \triangle$.

For a triangulation $\mathcal{T}$, let $\delta = \max_{\triangle \in \mathcal{T}} diam(\triangle)$. When considering a sequence $\{\mathcal{T}_j\}_{j=0}^{\infty} = \{\mathcal{T}_j\}$ of triangulations, we denote by δ_j the diameter of the triangulation $\mathcal{T}_j$. In the sequel, we deal with sequences $\{\mathcal{T}_j\}$ of triangulations satisfying the following conditions:

B1. For each $j \geq 0$, $\mathcal{T}_{j+1}$ is a refinement of $\mathcal{T}_j$, i.e., for each $\triangle \in \mathcal{T}_{j+1}$ there is $\tilde{\triangle} \in \mathcal{T}_j$ such that $\triangle \subset \tilde{\triangle}$.

B2. $\delta_j \to 0$ as $j \to \infty$.

B3. For $j \geq 0$, if v is a vertex in some $\triangle \in \mathcal{T}_j$, then v is a vertex in some $\triangle \in \mathcal{T}_{j+1}$.

For a simplex $\triangle$, we denote the set of vertices of $\triangle$ by $V(\triangle)$; for $v \in V(\triangle)$, let H_v be the $(n-1)$–dimensional hyperplane passing through the vertices of $\triangle$ other than that v and we put

$$\rho(\triangle) = \min_{v \in V(\triangle)} dist(v, H_v) \tag{5.10}$$

Now, consider admissible sequence of triangulations, recall (5.7).

Definition 5.4.2 *Let $\{\mathcal{T}_j\}$ be a sequence of triangulations satisfying B1. Then, $\{\mathcal{T}_j\}$ is an admissible sequence of triangulations if the following conditions hold:*

T1. There is a constant $c_2 > 0$, independent of j, such that,

$$c_2^{-1} 2^{-js} \leq \delta_j \leq c_2 2^{-js}, \quad s \in \mathbb{R}.$$

T2. There are constants $0 < c_3 < c_4 < 1$ such that, for all $j \geq 0$,

$$c_3 \delta_j \leq \delta_{j+1} \leq c_4 \delta_j,$$

where δ_j is the diameter of the triangulation $\mathcal{T}_j$.

T3. There is a constant $c_5 > 0$, independent of j, such that, for all $\triangle \in \mathcal{T}_j$,

$$\rho(\triangle) \geq c_5 diam(\triangle),$$

where $\rho(\triangle)$ is defined by (5.10).

T4. There exist constants $a > 0$ and $b > 0$, independent of j, such that if $x \in \triangle\mathcal{T}_j$, $y \in \triangle'\mathcal{T}_j$ and $|x-y| \leq a\delta_j$, then there is a point $z \in \triangle\cap\triangle'$ such that $|z-x| \leq b|x-y|$ and $|z-y| \leq b|x-y|$.

Note that T1 implies that $\delta_j/c_2 \leq diam(\triangle) \leq \delta_j$ for $\triangle \in \mathcal{T}_j$, so from T3 one obtains

$$\rho(\triangle) \geq \frac{c_5}{c_2}\delta_j, \quad for\ \triangle \in \mathcal{T}_j.$$

Condition T3 means that the simplices are not too flat, while condition T4 can be interpreted as requiring that the spaces between simplices are not too flat. In case of triangulations of **general closed sets**, condition T4 must be included in the assumptions.

Let ξ be a vertex of a simplex $\triangle$, and let ϕ_ξ be the **interpolating function** which is equal to 1 at ξ and equal to 0 at the other vertices of $\triangle$. It should be clear that

$$\max_{\xi\in V(\triangle)} |\nabla\phi_\xi| = 1/\rho(\triangle), \tag{5.11}$$

where $|\nabla\phi_\xi|$ denotes the gradient of ϕ_ξ.

5.4.2 Basis Functions : The General Setting

Let $\mathcal{T} = \bigcup_{m\in\mathbb{Z}} \mathcal{T}_m$ be a admissible sequence of triangulation. For $m \in \mathbb{Z}$, and $k \geq 1$, we denote by $\mathcal{S}_m^k = \mathcal{S}^k(\mathcal{T}_m)$ the set of all polynomial functions of degree $< k$ over $\mathcal{T}_m$; i.e., $s \in \mathcal{S}_m^k$ if and only if $s = \sum_{\triangle\in\mathcal{T}_m} \mathbb{1}_\triangle \cdot P_\triangle$, where $\mathbb{1}_\triangle$ is the characteristic functions of $\triangle$ and $P_\triangle \in \Pi_k$.

We assume that for each $m \in \mathbb{Z}$ there is a subspace $\mathcal{S}_m$ of $\mathcal{S}_m^k$ and a family $\Phi_m = \{\varphi_\theta : \theta \in \Theta_m\} \subset \mathcal{S}_m$ of basis functions satisfying the following conditions:

1. $\Pi_k \subset \mathcal{S}_m$ for some $1 \leq \tilde{k} \leq k$ ($\tilde{k}$ independent of m).

2. $\mathcal{S}_m \subset \mathcal{S}_{m+1}$ ($m \in \mathbb{Z}$).

3. For any $s \in \mathcal{S}_m$ there exist a unique sequence of real coefficients $a(s) = (a_\theta(s))_{\theta\in\Theta_m}$ such that

$$s = \sum_{\theta\in\Theta_m} a_\theta(s)\varphi_\theta.$$

(Thus, Φ_m is a basis for $\mathcal{S}_m$ and $(a_\theta(\cdot))_{\theta\in\Theta_m}$ are dual functionals.)

4. For each $\theta \in \Theta_m$ there is a vertex $v = v_\theta \in \mathcal{V}_m$ such that

$$supp\varphi_\theta \subset \triangle^* := E_\theta, \tag{5.12}$$

$$\|\varphi_\theta\|_{L_\infty(\mathbb{R}^d)} = \|\varphi_\theta\|_{L_\infty(E_\theta)} \leq M_1, \tag{5.13}$$

$$|a_\theta(s)| \leq M_2\|s\|_{L_\infty(E_\theta)}, \tag{5.14}$$

where M_1, and M_2 are positive constants, all independent of θ and m.

We denote $\mathcal{S} = \mathcal{S}_{m\in\mathbb{Z}}$, $\Phi := \bigcup_{m\in\mathbb{Z}} \Phi_m$ and $\Theta := \bigcup_{m\in\mathbb{Z}} \Theta_m$. We shall call $\mathcal{S}$ is spline multiresolution over $\mathcal{T}$ with family of basis function Φ.

A simple example of spline multiresolution is the sequence $(\mathcal{S}_m)_{m\in\mathbb{Z}}$ of all continuous piecewise linear functions ($r = 0$, $k = 2$) on the levels $(\mathcal{T}_m)_{m\in\mathbb{Z}}$ of a given regular-triangulation $\mathcal{T}$ of $\mathbb{R}^d$. A basis for each space $\mathcal{S}_m$ is given by the set Φ_m of the Courant elements φ_θ supported on the cells θ of $\mathcal{T}_m$ (θ is the union of all triangles of $\mathcal{T}_m$ attached to a vertex, say v_θ). The function φ_θ takes the value 1 at v_θ and the value 0 at all other vertices. Box splines with the corresponding ladder of spline spaces provide another example of a spline multiresolution. Concrete constructions of infinitely differentiable basis functions associated with spline multiresolution over regular triangulations will be discussed in next section.

Note that Θ and Θ_m ($m \in \mathbb{Z}$) above are simply index sets, in the case of Courant elements, can be identified as sets of cells (supports of basis functions). In general, several basis functions of Φ_m may have the same support. However, the supports of only constant of them may overlap.

It follows from the above conditions that each basis Φ_m is L_q-stable for all $0 < q \leq \infty$, i.e., if $g := \sum_{\theta\in\Theta_m} b_\theta\varphi_\theta$, where $\{b_\theta\}_{\theta\in\Theta_m}$ is an arbitrary sequence of real numbers, then

$$\|g\|_q \approx (\sum_{\theta\in\Theta_m} \|b_\theta\varphi_\theta\|_q^q)^{1/q}$$

with constants of equivalence independent of m and g.

5.4.3 Quasi–interpolant

For $0 < q \le \infty$ and an arbitrary triangle $\triangle$, we let $P_{\triangle,q} : L_q(\triangle) \to \Pi_k$ be a projector such that

$$\|f - P_{\triangle,q}\| \le cE_k(f,\triangle)_q \quad for\ f \in L_q(\triangle). \tag{5.15}$$

Note that $P_{\triangle,q}$ can be realized as a linear operator if $q \ge 1$. If $0 < q < 1$ we would be able to construct non–zero bounded linear functional on L_q.

Define a linear operator $Q_m : \mathcal{S}(\mathcal{T}_m) \to \mathcal{S}_m$ as follows: for each $\theta \in \Theta_m$ let $\lambda_\theta : \mathcal{S}(\mathcal{T}_m) \to \mathbb{R}$ be a linear functional such that

$$\lambda_\theta(s|E_\theta) = a_\theta(s), \quad s \in \mathcal{S}_m, \quad and$$

$$|\lambda_\theta(f)| \le M_2\|f\|_{L_\infty(E_\theta)}, \quad f \in \mathcal{S}^k(\mathcal{T}_m)|_{E_\theta}.$$

Such linear functional always exists by Hahn–Banach theorem. We set

$$Q_m(s) := \sum_{\theta\in\Theta_m} \lambda_\theta(s|E_\theta)\varphi_\theta, \quad s \in \mathcal{S}^k(\mathcal{T}_m)|_{E_\theta}.$$

Clearly $Q_m(s) = s$ if $s \in \mathcal{S}_m$, and thus Q_m is a linear projector of $\mathcal{S}(\mathcal{T}_m)$ onto $\mathcal{S}_m$. Moreover, Q_m is a bounded projector. For any $s \in \mathcal{S}(\mathcal{T}_m)$, $0 < q \le \infty$, and $\triangle \in \mathcal{T}_m$

$$\|Q_m(s)\|_{L_q(\triangle)} \le c\|s\|_{L_q(\Omega^\ell_\triangle)},$$

with a constant c independent of m, $\triangle$, and s.

We denote $\mathcal{S}_{-\infty} := \bigcap_{m\in\mathbb{Z}} \mathcal{S}_m$. Note that if $s \in \mathcal{S}_{-\infty}$, $s \ne$ constant , and $|\{x \in \mathbb{R}^d : |s(x)| > t\}| < \infty$ for some $t > 0$, then $s \equiv 0$ and, in particular, if $s \in \mathcal{S}_{-\infty} \cap L_p$ $(p < \infty)$, then $s \equiv 0$.

Triangular Function. Clearly, the basis function ϕ defines a triangular function on compact set F_j for each resolution level $j = 0, 1, 2, \cdots$. Denote by $\mathcal{U}_j$ be the set of all vertices of the triangulation $\mathcal{T}_j$, and for $\xi \in \mathcal{U}_j$, let $s_j(\xi)$ be the family of simplices from $\mathcal{T}_j$ having ξ as one of the vertices, and let $F_j(\xi)$ be the union of these simplices, i.e., let

$$s_j(\xi) = \{\triangle \in \mathcal{T}_j :\ \xi \in V(\triangle)\}, \quad F_j(\xi) = \bigcup_{\triangle\in s_j(\xi)} \triangle. \tag{5.16}$$

For $\xi \in \mathcal{U}_j$, let $\phi_{j,\xi}$ be the unique function defined on $F_j = \bigcup_{\triangle \in \mathcal{T}_j} \triangle$ such that

$$\phi_{j,\xi} = \delta_{\xi,\eta}, \quad for\ all\ \eta \in \mathcal{U}_j.$$

$$\mathcal{V}_0 = \mathcal{U}_0, \quad \mathcal{V}_i = \mathcal{U}_i \backslash \mathcal{U}_{i-1}, \quad for\ i > 0, \tag{5.17}$$

and for latter convenience, let $\mathcal{V} := \cup_{i=0}^{\infty} \mathcal{V}_i$. Note that the support of $\phi_{j,\xi}$ is $F_j(\xi)$. Condition T3 and T4 guarantee that the function $\phi_{j,\xi}$ satisfy a Lipschitz constant depending only on δ_j. We introduce one more condition on a sequence $\{\mathcal{T}_j\}$ of triangulation of a compact set F. Recall that $\mathcal{U}_j$ denote the set of vertices of $\{\mathcal{T}_j\}$ and let

T5. there is a constant c_6 such that for all $j \geq 0$, $\xi \in \mathcal{U}_j$ and $x, y \in F_j$, we have

$$|\phi_{j,\xi}(x) - \phi_{j,\xi}(y)| \leq c_6 |x - y| / \delta_j.$$

In the sequel, functions $\phi_{j,\xi}$ are extended to Lipschitz function on $\mathbb{R}^d$, where $d > 0$ is odd, since we need the polynomial reproduction property of the basis function expansion.

Proposition 5.4.1 *Let $\{\mathcal{T}_j\}$ be a an admissible sequence of triangulation of a compact set $F \subset \mathbb{R}^d$. Then there are constants c_7 and r, independent of j, such that each function $\phi_{j,\xi}$ can be extended to a function $\tilde{\phi}_{j,\xi}$ on $\mathbb{R}^d$, satisfying*

$$supp\tilde{\phi}_{j,\xi} \leq B(\xi, r\delta_j), \quad \|\tilde{\phi}_{j,\xi}\|_\infty \leq 1,$$

and

$$|\tilde{\phi}_{j,\xi}(x) - \tilde{\phi}_{j,\xi}(y)| \leq c_7 |x - y| / \delta_j, \quad for\ x, y \in \mathbb{R}^d.$$

Proof : Let $\gamma > 0$, and put $E = \{x : d(x, F_j(\xi)) \geq \gamma\delta_j\}$. Extend the domain of $\phi_{j,\xi}$ to include E by defining it as zero there. Then, if $x \in F_j(\xi)$ and $y \in E$, we have

$$\phi_{j,\xi}(x) - \phi_{j,\xi}(y) \leq 1 \leq |x - y| / (\gamma\delta_j).$$

Clearly, ϕ is a Lipschitz function on $F_j \cup E$. The Whitney extension theorem yields an extension $\tilde{\phi}$ of ϕ to $\mathbb{R}^n$ which has the desired properties. ∎

The extension theorem gives, in fact, that $0 \leq \tilde{\phi}_{j,\xi} \leq 1$. Then we have the following lemma

.

Lemma 5.4.2 *Let F be a compact subset of $\mathbb{R}^d$, and $\{\mathcal{T}_j\}$ be a an admissible sequence of triangulation of F satisfying the above triangulation conditions. For $\xi \in \mathcal{U}_j$, $\tilde{\phi}_{j,\xi}$ is the unique function on $F_j = \bigcup_{\triangle^* \in \mathcal{T}_j} \triangle^*$ such that $\tilde{\phi}_{j,\xi} = \delta_{\xi,\eta}$ for all $\eta \in \mathcal{U}_j$ on each simplex $\triangle^* \in \mathcal{T}_j$. Then there is a unique function $s \in \mathcal{S}_j$ such that*

$$\mathcal{S}_j = \{s : s = \sum_{\xi \in \mathcal{U}_j} a_\xi \tilde{\phi}_{j,\xi},\ a_\xi \in \mathbb{R}\}, \tag{5.18}$$

where $\mathcal{S}_j$ is the space of functions generated by $\{\tilde{\phi}_{j,\xi} : \xi \in \mathcal{U}_j\}$, whose restriction on F_j are linear on each $\triangle \in \mathcal{T}_j$. Then, for each $\xi \in \mathcal{U}_j$,

$$supp\ \tilde{\phi}_\xi \subset \triangle^* =: E_\xi,$$

$$\|\tilde{\phi}_\xi\|_{L_\infty(\mathbb{R}^d)} = \|\tilde{\phi}_\xi\|_{L_\infty(E_\xi)} \le M_1,$$

$$|a_\xi| \le M_2 \|s\|_{L_\infty(E_\xi)}, \quad s \in \mathcal{S}^k(\mathcal{T}_j),$$

where M_1, M_2 are positive constants depending only on k, r, δ_i and j. Moreover, for each $\triangle \in \mathcal{T}_j$,

$$\|s|_\triangle\|_{L_\infty(\triangle^*)} \le c\delta_j \max_{\xi \in \mathcal{U}_j} |a_\xi|, \tag{5.19}$$

where c is a constant depending only on j and $\mathcal{U}_j$.

Proof : Please see Lemma 2.4 of Davydov and Petrushev (2003). ∎

Further, (5.19) implies that $\|\tilde{\phi}_{j,\xi}\| \le c\delta_j$ and supp $\tilde{\phi}_{j,\xi}$ is contained in $\triangle$ whenever $\xi \in \mathcal{U}_j$. Also, by Markov's inequality,

$$|a_\xi| \le C\delta_j \|s|_{\triangle^*}\|_{L_\infty(\triangle^*)}. \tag{5.20}$$

Thus, we showed that $\Phi_j = \{\tilde{\phi}_\xi : \xi \in \mathcal{U}_j\}$ satisfies all the requirements of Section 5.4.2 with $\mathcal{S}_j = \mathcal{S}^k(\mathcal{T}_j)$ and $\tilde{k} = k$. Obviously, $\Pi_k \subset \mathcal{S}_j$.

Consider the space $\mathcal{S}_j$ consists of the restriction of F of functions which are defined and continuous on F_j, and linear on each $\triangle \in \mathcal{T}_j$. As

$$\tilde{\phi}_{j,\xi} = \sum_{\eta \in F_j(\xi) \cap \mathcal{U}_{j+1}} \tilde{\phi}_{j,\xi}(\eta) \tilde{\phi}_{j+1,\xi},$$

then we have $\mathcal{S}_j \subset \mathcal{S}_{j+1}$. The linear functional $\lambda_\eta : \mathcal{S}^k(\mathcal{T}_j) \cap L_\infty(E_\eta) \to \mathbb{R}$, $\eta \in \mathcal{U}_j$, with properties

$$\lambda_\eta(s|_{E_\eta}) = \eta(s), \quad s \in \mathcal{S}^k(\mathcal{T}_j);$$

$$|\lambda_\eta(f)| \leq M_2 \|f\|_{L_\infty(E_\eta)}, \quad f \in \mathcal{S}^k(\mathcal{T}_j) \cap L_\infty(E_\eta);$$

needed in the definition of Q_j, can be defined in a construction manner as:

$$\lambda_\eta(f) = \begin{cases} f(\eta) & \text{for } \eta \in \mathcal{U}_0 \\ f(\eta) - S_{k-1} f(\eta) & \text{for } \eta \in \mathcal{U}_j \ j > 0, \end{cases}$$

where

$$Q_j f = \sum_{\theta \in K} a_\theta(f) \tilde{\phi}_\eta, \quad K \textit{ is compact subset of } \eta.$$

The above lemma gives us that the linear operators Q_j, $j = 0, 1, 2, \cdots$, with the following properties:

Proposition 5.4.2 *Suppose $f \in C(F)$, then we have,*

$$Q_j f = \sum_{\xi \in \mathcal{U}_j} c_\xi(f) \tilde{\phi}_\xi, \quad j = 0, 1, 2, \cdots,$$

where $c_\xi(f)$ is a bounded linear functional on a compact set $K \subset \mathbb{R}^d$, $d > 0$ is odd. Q_j is a kind of nonorthogonal projection onto $\Pi_k \subset \mathcal{S}_j$, where Π_k is a polynomial space with degree less than or equal to k. In particular, $Q_j q = q$, for $q \in \Pi_k$, for all $j = 0, 1, 2, \cdots$.

Proof : By the above Lemma 5.4.2 and the following discussions. ∎

5.4.4 Sequence of Linear Operators

Our goal in this section is to employ families $T = (T_j)_{j \in \mathbb{N}}$ of linear operators in construction of wavelets. We assume that we have in hand a sequence of linear operators T_j, $j \in \mathbb{N}$, given by

$$T_j f = \sum_{\xi \in \mathcal{U}_j} c_{j,\xi}(f) \phi_{j,\xi}, \quad j = 0, 1, 2, \cdots, \tag{5.21}$$

where $c_{j,\xi}(f)$ is a bounded linear functional on a compact set $K \subset \mathbb{R}^d$, $d > 0$ is odd. It is noted that the biorthogonal wavelets are typical candidates for the function $\phi_{j,\xi}$ for each $\xi \in \mathcal{U}_j$ and for $j = 0, 1, \cdots$, which satisfy the following properties:

C1. supp $\phi \subset [-L, L]^d$, $L \in \mathbb{N}$,

The integer translates of ϕ give rise to a linear operator T_0 defined on $f \in C(\Omega)$, by

$$T_0 f := \sum_{\xi \in \mathcal{U}_0} c_\xi(f)\phi_\xi,$$

while scaling and translation yields a sequence $T = (T_j)_{j\in\mathbb{N}}$ of operators given by

$$T_j f := \sum_{\xi \in \mathcal{U}_j} c_{j,\xi}(f)\phi_{j,\xi}, \quad j = 1, 2, \cdots .$$

C2. We assume that the sequence $(T_j)_{j\in\mathbb{N}}$ has the following properties: For some $d > 0$

$$T_j p = p \quad p \in \Pi_d.$$

C3. For every $j,\ v\ \in\ \mathbb{N}$ with $j \geq v$

$$T_j\ T_v = T_v.$$

C4. $\phi_j \in C_0^\infty$.

C5. In addition, for each $j \in \mathbb{N}$ we will often needed the ϕ_j is the basis over a compact subset of Ω; by this we mean that the family of functions

$$\{\phi_j : j \in \Gamma_j \quad \textit{is not identically zero on } \Omega\}$$

is linearly independent over Ω and the space spanned by the function ϕ_j is finite dimensional.

All these assumption are standard and well understood because, these kind of operators play a dominant role in the characterization of the approximation orders of shift–invariant spaces and the construction of interpolating operators. For instance, (C2) is usually related to the approximation properties of the sequence $(T_j)_{j\in\mathbb{N}}$ and holds if ϕ satisfies the conditions of if $\varphi(r) = (r^2 + \gamma^2)^{\beta/2}$ and d is odd, then there are finite set $N \subset \mathbb{Z}^d$ and real coefficients $\{\mu_k\}_{k\in N}$ such that T and ϕ defined as above satisfy

$$|\phi(x)| = O(\|x\|^{-2n-1}) \quad as\ \|x\| \to \infty,$$

and

$$Tp = p, \quad for\ all\ p \in \Pi_n,\ n \leq d.$$

The property (C5) is readily available with the interpolation function on a compact set Ω. On the other hand, we would show that the rest of the assumptions are always satisfied within the framework of interpolating wavelet basis of multiquadric functions. Now we have to show that the above interpolating functions described above posses sufficient conditions for the assumptions C1 and C3 to hold.

5.5 Bounded Domain

In this section, construction of domains $\Omega_j \subset \mathbb{R}^d$ and that these domains Ω_j form a partition of Ω_j, i.e., collection $\mathcal{C}_j$ of cells which satisfy the conditions of multiresolution. Thus, for these domains, the construction of the multiresolution spaces $V_j(\Omega)$ and all the ensuant properties of multiresolution are hold.

Assume that Ω is a bounded, simply connected domain (i.e. an open set) satisfying the uniform cone condition. We recall that the uniform cone condition means that there is an open cone K with vertex at the origin such that for each point x on the boundary of a suitable translate and rotation K' of K has vertex x and $K' \cap B(x,r) \subseteq \Omega$ for some ball $B(x,r)$ centered at x with radius r.

Let $\Omega \subset \mathbb{R}^d$ be a compact set and let $f \in L_q(\Omega)$, $0 < q \leq \infty$ ($f \in C$ if $q = \infty$) be a function decomposed with normal multiresolution approximation with uniform parameterization. Then, for each $j = 0, 1, \cdots$, $(x_{j,k})_{k \in \mathbb{N}_0}$ be a strictly increasing sequence of centers in Ω which define

$$\Omega_j = \{2^{-j}k : k \in \mathbb{Z}^d, \Omega \cap 2^{-j}(k + [-L, L]^d) \neq \emptyset\}$$

denote the lattice compact sets for which the support of $\phi_k = \phi(2^j \cdot -k)$ intersect Ω.

To describe the construction of multiresolution on bounded domain similar to the description by Cohen et al. (2000). In that paper, they have constructed a ladder of space $\mathcal{S}_j(\Omega)$, $j = 0, 1, \cdots$, which retains the important properties of multiresolution.

Since the support of ϕ is contained in the cube $[-L, L]^d$, it is obvious that if $2^{-j}k \notin \Omega_j$ then $\mathrm{supp}\phi(2^j \cdot -k) \cap \Omega = \emptyset$. Therefore, we need only to consider k, j's with $2^{-j}k \in \Omega_j$,

$j \in \mathbb{N}$.

To simplify our notation, for each lattice point $\gamma = 2^{-j}k$ of Ω_j, we write,ϕ_γ instead of $2^{jd}\phi(2^j \cdot -k)$. However, since a lattice point may belong to different Ω_j's we will always correlate ϕ_γ with a specific dyadic level j which we make clear in all instances.

The construction of $\mathcal{S}_j(\Omega)$, $j = 0, 1, \cdots$, proceeds by partitioning Ω_j into a family $\mathcal{C}_j$ of disjoint subsets (cells) C of Ω_j. In other words, each $\mathcal{C}_j$ consists of a collection of disjoint (cells) $C \in \mathcal{C}_j$ such that $\bigcup_{C \in \mathcal{C}_j} C = \Omega_j$.

Of course, not all bounded domains Ω will admit a multiresolution analysis $\mathcal{S}_j(\Omega)$, $j \in \mathbb{N}$. The admissibility of Ω depends foremost on the properties of the cells C in $\mathcal{C}_j$. We will briefly recall the notation of Cohen et al. (2000) and describe the properties imposed on the cells in Cohen et al. (2000) that guarantee the existence of $\mathcal{S}_j(\Omega)$, $j \in \mathbb{N}$.

We assume that each $\mathcal{C}_j$ can be partitioned into sub collections $\mathcal{C}_j(I, \gamma)$ where $I \subset [1, \cdots, d]$ and $\gamma = (\gamma_l)_{l \in I}$, $(\gamma_l)_{l \in I} = (\sigma_i \cdot \sigma_j)_{i,j \in I} \subset \{-1, 1\}^{|I|}$, where $(\sigma_i \cdot \sigma_j)_{i,j \in I}$ is dot product in Euclidean space $\mathbb{R}^d$, i.e.,

$$\mathcal{C}_j = \bigcup_{I,\gamma} \mathcal{C}_j(I, \gamma),$$

where

$$\mathcal{C}_j(I, \gamma) \cap \mathcal{C}_j(I', \gamma') = \emptyset \quad for\ (I, \gamma) \neq (I', \gamma').$$

Moreover, each cell $C \in \mathcal{C}_j(I, \gamma)$ is of the form

$$C = k + D(k),$$

with $k \in \Omega_j$ a lattice point (called the representer of C) and

$$D(k) \subset span\{e_i : i \in I\} \bigcap 2^{-j}\mathbb{Z}^d,$$

with e_i , $i = 1, \cdots, d$, the coordinate vectors in $\mathbb{R}^d$.

For each cell C and its representer k, we define

$$G(C) := \{k + 2^{-j}T_\gamma \alpha : \alpha \in \mathbb{Z}^d_+, \quad 0 \leq \alpha \leq N\},$$

where for a sequence γ the transformation T_γ is defined on $\mathbb{R}^d$ by

$$T_\gamma \left(\sum_{i \in I} \lambda_i e_i \right) := \sum_{i \in I} \gamma_i \lambda_i e_i.$$

In other words $G(C)$ consists of a square array of $(N+1)^{|I|}$ lattice points emanating from k and expanded in the direction defined by I and γ.

Also, for a set $K = [k_1, \cdots, k_m] \subset [1, \cdots, d]$ with $k_1 < k_2 < \cdots < k_m$, and a point $x \in \mathbb{R}^d$, we define x_K to be the point whose coordinates are those of x corresponding to the indices of K, i.e., $x_K := (x_{k_1}, \cdots, x_{k_m})$. If $C \in \mathcal{C}_j(I, \gamma)$ is a cell in $\mathcal{C}_j$, we let $\Lambda(C)$ be the set of all $\alpha \in \Lambda$ for which $\alpha_j = 0$, $j \in \{1, \cdots, d\} \backslash I$.

Further assumptions have to be made on the cells that guarantee the nestedness of the sequence $\mathcal{S}_j(\Omega)$, $j \in \mathbb{N}$. The following two conditions will ensure the nestedness of the multiresolution approximation spaces.

M1. If $C \in \mathcal{C}_j(I, \gamma)$ and $C^{'} \in \mathcal{C}_{j+1}(I^{'}, \gamma^{'})$ satisfy

$$[C] \cap C^{'} \neq \emptyset,$$

then

$$I^{'} \subseteq I. \tag{5.22}$$

M2. If $C \in \mathcal{C}_j(I, \gamma)$ and $C^{'} \in \mathcal{C}_j(I^{'}, \gamma^{'})$ are two cells from $\mathcal{C}_j$ with $C \neq C^{'}$ and

$$[C, I] \cap [C^{'}, I^{'}] \neq \emptyset,$$

then

$$I^{'} \subset I, \quad I^{'} \neq I. \tag{5.23}$$

M3. Finally, it will be important to ensure that all the basis functions have small support. There exists a constant M such that

$$diam[C] \leq M 2^{-js}, \quad C \in \mathcal{C}_j, \quad s \in \mathbb{R}. \tag{5.24}$$

5.5.1 Normal Interpolating Wavelet Basis in $C(F)$

Now, we describe quasi–interpolating wavelet basis in $C(F)$. For $\xi \in \mathcal{V}_i$, let $\psi_{i,\xi} := \widetilde{\phi_{i,\xi}}$; if $\xi \in \mathcal{V}_i$ with $i > 0$, then put

$$\triangle(\xi) = \mathcal{U}_{i-1} \cap \bigcap \{\triangle : \triangle \in \mathcal{T}_{i-1}, \xi \in \triangle\}.$$

Let $f \in C(F)$, and suppose that f is decomposed with normal multiresolution approximation into regular sequence of triangles $\{\mathcal{T}_j\}_{j\geq 0}$ satisfying the properties B1 and B2. Then the vertical offset coefficients are computed as follows:

$$c_{i,\xi}(f) = \begin{cases} f(\xi) & \text{for } \xi \in \mathcal{V}_0, \\ f(\xi) - \sum_{\eta\in\triangle(\xi)} q_{\xi,\eta} f(\eta) & \text{for } \xi \in \mathcal{V}_i \text{ with } i > 0, \end{cases} \tag{5.25}$$

where $\xi = \sum_{\eta\in\triangle(\xi)} q_{\xi,\eta}\eta$.

Lemma 5.5.1 *For each ξ and $\eta \in \triangle(\xi)$, $q_{\xi,\eta} > 0$ and $\sum_{\eta\in\triangle(\xi)} q_{\xi,\eta} = 1$. The coefficients $c_{i,\xi}(f)$ are chosen in such a way that*

$$f(\eta) = \sum_{i=0}^{j} \sum_{\xi\in\mathcal{V}_i} c_{i,\xi}(f)\psi_{i,\xi}(\eta), \quad \textit{for each } \eta \in \mathcal{U}_i,\ i \geq 0. \tag{5.26}$$

Proof : It can be easily proved by induction with respect to i. ∎

As a consequence of Lemma 5.5.1 we state the following proposition.

Proposition 5.5.1 *Let F be a compact set in $\mathbb{R}^d$, $d > 0$ and let admissible sequence of triangles $\{\mathcal{T}_j\}_{j\geq 0}$ satisfying the properties B1 and B2. Then the system of functions $\{\psi_\xi : \xi \in \mathcal{V}\}$ satisfying (5.25) is a basis in $C(F)$. More precisely, for each $f \in C(F)$, we have*

$$f = \sum_{i\in\mathbb{N}_0} \sum_{\xi\in\mathcal{V}_i} c_{i,\xi}(f)\psi_{i,\xi},$$

with the series uniformly convergent on F, and coefficients $c_{i,\xi}(f)$ are given by (5.25).

Proof : It is proved analogously as Proposition 3.3 in Ryll (1973) (where the case of a cube is considered,), so the details of the proof of Proposition 5.5.1 could be obtained from Ryll (1973). ∎

Suppose that $\mathcal{T}$ is a admissible sequence of triangulation then, since by the property; any two norms on a finite dimensional space are equivalent, we have that there are two positive constants C_1 and C_2 such that

$$C_1(2^{-js})^{d/p-1} \leq \|\tilde{\phi}_{j,\xi}\|_p \leq C_2(2^{-js})^{d/p-1}, \quad s \in \mathbb{R} \textit{ for } 1 \leq p \leq \infty. \tag{5.27}$$

Then we have the following theorem.

Theorem 5.5.1 *For* $1 \leq p \leq \infty$, $\{(2^{js})^{1-d/p}\tilde{\phi}_{j,\xi} : \xi \in \mathcal{U}_j\}$ *forms a uniformly stable basis for any* $f \in L_p(\Omega)$ *can be represented as:*

$$f = \sum_{\xi \in \mathcal{U}_j} a_\xi \tilde{\phi}_{j,\xi}, \quad for\ some\ a_\xi \in \mathbb{R},$$

and

$$A(2^{js})^{1-d/p}\left(\sum_{\xi\in\mathcal{U}_j}|a_\xi|^p\right)^{1/p} \leq \|f\|_{p,\Omega} \leq B(2^{js})^{1-d/p}\left(\sum_{\xi\in\mathcal{U}_j}|a_\xi|^p\right)^{1/p}, \quad s \in \mathbb{R},$$

where A *and* B *are positive constants depend of* $\tilde{\phi}$ *and* d.

Proof : The results easily follows from 5.19, 5.20 and 5.27. ∎

5.6 Multiresolution Analysis

The construction of multiresolution analysis over triangulation is closely related to the construction of nested spline spaces. We are concerned with the interpolating basis constructed in previous section, and relate these basis functions to the fairly general definition of multiresolution analysis following the work of Dahmen (1991, 1994a,b).

In the previous section we have constructed a nested sequence of subspaces $\mathcal{S}_0 \subset \mathcal{S}_1 \subset \cdots$ and we introduced a linear operator Q_j. In this section we look for Banach space $\mathcal{F}$ such that sequence of subspaces $\{\mathcal{S}_j\}_{j\in\mathbb{N}}$ and the operators Q_j form a multiresolution analysis in the sense of following definition (Maes, and Bultheel, 2006).

Definition 5.6.1 *A multiresolution analysis consists of*

1. *A Banach space* $\mathcal{F}$ *of functions defined on a compact set* $\Omega \subset \mathbb{R}^d$, $d > 0$, *with associated norm* $\|\cdot\|_{\mathcal{F}}$.

2. *A nested sequence of closed subspaces* $\mathcal{S}_0 \subset \mathcal{S}_j \subset \cdots \subset \mathcal{F}$ *that are dense in* $\mathcal{F}$,

$$\overline{\bigcup_{j=0}^{\infty} \mathcal{S}_j} = \mathcal{F}.$$

3. A collection of local linear operator

$$Q_j : \mathcal{F} \to \mathcal{S}_j$$

with the properties

$$Q_j Q_j = Q_j,$$

$$Q_j(\mathcal{F}) = \mathcal{S}_j,$$

$$Q_{j+1} Q_j = Q_j,$$

for every integers $j \geq 0$.

With the projection operator Q_j given, we can define the multiscale decomposition of function $f \in \mathcal{F}$ as

$$f = \sum_{j \in \mathbb{N}_0 \cap -1} \sum_{\xi \in \mathcal{U}_j} d_{j,\xi} \psi_{j,\xi}, \tag{5.28}$$

where we have set for simplicity $\psi_{-1} := \phi_0$. We will refer to this function as wavelets, despite the fact the fact they do not have a vanishing moment. Because the space $\mathcal{S}_m$ are dense in $\mathcal{F}$, every function $f \in \mathcal{F}$ has a representation (5.28) with $n \to \infty$. The decomposition (5.28) is particularly useful if the norm of f, in some L_p spaces or Sobolev spaces, can be determined solely by examining the size of the coefficients $c_{j,\xi}(f)$. In particular, the multiscale basis forms a strongly stable basis for some L_p spaces or Sobolev spaces.

5.6.1 Banach Spaces with Sequence of Operators

A Banach space $\mathcal{F}$, consider the $C(F)$ space, of functions defined on compact set F that are continuous. The following propositions show that the operator Q_j defined in (5.6.1) admits multiresolution analysis over $C(F)$, in view of the Definition 5.6.1.

Proposition 5.6.1 *For each $j \geq 0$, we have*

$$Q_j Q_{j+1} = Q_j.$$

Proof : From the construction of Q_j, we have

$$Q_{j+1}f(\xi) = f(\xi), \quad \forall \xi \in \mathcal{V}_{j+1}.$$

Obviously, we have

$$Q_jQ_{j+1}f(\xi) = f(\xi), \quad \forall \xi \in \mathcal{V}_j \subset \mathcal{V}_{j+1},$$

and

$$Q_jf(\xi) = f(\xi), \quad \forall \xi \in \mathcal{V}_j \subset \mathcal{V}_{j+1}. \quad \blacksquare$$

From the uniqueness of the interpolation property we have $Q_jQ_{j+1} = Q_j$. $\blacksquare$

There are still two properties required for a multiresolution analysis to be proved, namely that the space $\overline{\bigcup_{j=0}^{\infty} \mathcal{S}_j}$ is dense in $C(F)$ and that the linear operator is uniformly bounded.

Proposition 5.6.2 *For every $C(F)$ and every $\xi \in F$ the following inequality holds*

$$|Q_jf(\xi)| \leq \|f\|_\infty.$$

Proof : We have

$$|Q_jf(\xi)| \leq \max |c_\xi(f)| \; \|\sum_{\xi \in F} \phi_\xi\|.$$

$$|Q_jf(\xi)| \leq \|f\|_\infty$$

holds. $\blacksquare$

The following proposition is the consequence of construction our sequence of linear operators Q_j, for all $j = 0, 1, \cdots$,

Proposition 5.6.3 *For every $f \in C(F)$*

$$\overline{\bigcup_{j=0}^{\infty} \mathcal{S}_j} = C(F).$$

Proof : In order to prove, we have to show that $\|f - Q_jf\| = 0$ as $j \to \infty$. For this, we assume that the sequence of operator $(Q_j)_{j \in \mathbb{N}}$ that satisfy the conditions (C1)–(C4) of subsection 5.4.4 for some $\alpha > 0$ (smoothness of function $f \in C^\alpha$ being interpolated). For any $j \in \mathbb{N}$, we define

$$\mathcal{Q}_L^j := \{Q \in \mathcal{Q} : (4L + 2\sqrt{d})^{-j-1} < \ell(Q) < (4L + 2\sqrt{d})^{-j}\}.$$

Moreover, for every $\eta \in \mathbb{R}^d$ and $j \in \mathbb{N}$, we denote by $Q_L^j \eta$ the cube centered at η with side length $4L + 2\sqrt{d}$, i.e.,

$$Q_L^j(\eta) := \eta + [-\sqrt{d} - 2L, \sqrt{d} + 2L]^d 2^{-j}.$$

Let $\eta \in \mathbb{R}^d$ and assume that $Q \in \mathcal{Q}^j$, $j \in \mathbb{N}$, contains η. It is easily seen that $Q \subset \eta + [-\sqrt{d}, \sqrt{d}]^d 2^{-j}$. Since supp $\tilde{\phi}_{j,\xi} \subset ([-l, L]^d + k)2^{-j}$ it follows that for every $\zeta \in Q$,

$$\begin{aligned} |Q_j f(\zeta)| &= |\sum_{k \in \mathbb{N}^d, \eta - k \in [-l, L]^d} 2^{jd} c_{j,\xi} \tilde{\phi}_{j,\xi}(\eta)| \\ &\lesssim \frac{1}{Q_L^j(\eta)} \int_{Q_L^j(\eta)} |f|. \end{aligned} \tag{5.29}$$

Employing (C2) it follows from the (5.29) that for any cube $Q \in \mathcal{Q}^j$, $j \in \mathbb{N}$, containing η,

$$\begin{aligned} |f - Q_j f(\eta)| &\le \|f - f_{Q_L^j(\eta)} + Q_j(f_{Q_L^j(\eta)}) - Q_j f\|_{L_\infty(Q)} \\ &\le \|f - f_{Q_L^j(\eta)}\|_{L_\infty(Q)} + \|Q_j(f - f_{Q_L^j(\eta)})\|_{L_\infty(Q)} \\ &\le \|f - f_{Q_L^j(\eta)}\|_{L_\infty(Q)} + \frac{1}{Q_L^j(\eta)} \int_{Q_L^j(\eta)} |f - f_{Q_L^j(\eta)}| \\ &\lesssim \|f - f_{Q_L^j(\eta)}\|_{L_\infty(Q)}, \end{aligned} \tag{5.30}$$

where $f_{Q_L^j(\eta)} := |Q_L^j(\eta)|^{-1} \int_{Q_L^j(\eta)} |f|$. From (5.30) it is easily seen that for every continuous function f,

$$\lim_{j \to \infty} Q_j f(x) = f(x).$$

Then we deduce the result. ∎

Remark 5.6.1 *The basis $\{\psi_\xi : \xi \in \mathcal{V}\}$ should be regarded as an analogue on F of the (R, D) interpolating wavelet with regard to Donoho (1992).*

5.7 Summary

It is shown that the sequence of operators Q_j, for all $j = 0, 1, \cdots$, admit the construction of multiresolution analysis in view of the concepts given in Section 5.4.4 which are non–orthogonal interpolating operators. The wavelets are defined as scale invariant compactly supported function using linear combination of multiquadric functions.

We thus have a wavelet decomposition which exhibits the $(c_\xi(f))$ explicitly as measures of error in approximation by $\mathcal{S}_j$, and which reconstructs continuous functions. This setting exactly fit into second generation setting, in particular, normal multiresolution wavelet (Daubechies et al., 1999) which is the variant of lifting scheme (Sweldens, 1996, 1997).

This function can be decomposed with normal wavelet decomposition. Now, these functions have to be characterized in Horizon class of functions, i.e., gray–scale images. Since, by the property of embedding of $C(F)$ as $B^\alpha_{p,q} \subset B^\alpha_{p,\infty} \subset B^\alpha_{p,1} \subset C(F)$, for $1 \leq p \leq \infty$, $1 \leq p < \infty$ with $1/p < \alpha < \infty$, the following chapter is devoted to the above decomposition of functions in less smooth spaces such as Besov spaces.

Chapter 6

Decomposition of Normal Wavelet into Function Spaces

This chapter is concerned with defining normal wavelet basis functions which decompose a function f in the framework of some known function spaces. In particular normal wavelet decomposition of a function f is a quasi–interpolating decomposition into a certain function spaces, such as Sobolev spaces, Besov spaces etc. On the other hand it leads to certain characterizations and some assertions concerning to Schauder bases for the function spaces. Specifically, isomorphism of spaces of function satisfying certain regularity conditions with space of sequences of real numbers have proved to be useful tool, e.g., **image compression**, etc. The idea of constructing such isomorphism is that the coefficients of Faber–Schauder series gives a linear isomorphism of the spaces of Hölder functions on $[0, 1]$ with exponent α, $0 < \alpha < 1$, and the space of bounded sequences. Later, results of this type have been obtained for spaces of functions satisfying the Hölder condition in the L_p–norm and for Besov spaces, also for multivariate functions and for functions defined on smooth compact manifolds, and for various spline basis, including the classical Franklin system (Ciesielski, 1963, 1966; Ciesielski et al., 1993). Analogous characterizations are also known for function spaces on $\mathbb{R}^n$ and wavelet bases (Frazier et al., 1991; Meyer, 1992; Wojtaszczyk, 1997).

Let Ω be a bounded simply connected subset of $\mathbb{R}^n$. Let f be a function in $L_p(\Omega)$,

$1 \leq p < \infty$ and $f \in C(\Omega)$ when $p = \infty$, is decomposed with normal multiresolution approximation then we have a finite set of n–dimensional simplices $\triangle$ covering Ω, such that all vertices of $\triangle$ are in $\partial\Omega$, and if two different $\triangle \in \mathcal{T}$ intersect, where $\mathcal{T}$ is the union of all $\triangle$ in Ω, then their intersection is a common face of lower dimension. In the sequel, we assume that the set Ω admits a admissible sequence of triangulations $\{\mathcal{T}_j\}_{j=0}^{\infty}$, and denote the set of vertices of $\mathcal{T}_j$ by $\mathcal{U}_j$. Define for $\xi \in \mathcal{U}_j$, a function $\phi_{j,\xi}$ on the union of all $\triangle \in \mathcal{T}_j$ as the piecewise linear function (i.e., piecewise polynomial of total degree $\leq n$) which is 1 at ξ and 0 at the other vertices of $\mathcal{U}_j$. Choosing at each level the functions corresponding to the newly added vertices, we arrive to a system of functions $\{\psi_\xi\}$ which can be regarded as an analogue on Ω of the Schauder basis. The system $\{\psi_\xi\}$ ordered in a natural way is an interpolating basis in the space $C(F)$ of continuous functions f on Ω, and we characterize Besov spaces $B_{p,q}^{\alpha}(\Omega)$ on Ω with $d/p < \alpha < 1$.

6.1 Images in Less Smooth Spaces

By remark 2.4.1 in Chapter 2, we have some sequence of interpolating functions $f_j(x)$ are Horizon class and with minimally smooth boundary. Let's introduce two notations. Consider the following very simple *horizon* model. Suppose there is a function $H(x)$, called the horizon, defined on the interval $[0, 1]$, and that the function is of the form of

$$f(x_1, x_2) = 1_{\{x_2 \geq H(x_1)\}}.$$

This models a black–and–white images with a horizon, where the image is white above the horizon and black below. We are interested in cases where the horizon is regular, and to measure this we use Hölder conditions. For $0 < \alpha \leq 1$ we say that $H \in HOLDER^{\alpha}(C)$ if

$$|H(x) - H(x^{'})| \leq C|x - x^{'}|^{\alpha}, \quad 0 \leq x, x^{'} \leq 1.$$

For $1 < \alpha \leq 2$ we say that $H \in HOLDER^{\alpha}(C)$ if

$$|H^{'}(x) - H^{'}(x^{'})| \leq C|x - x^{'}|^{(\alpha-1)}, \quad 0 \leq x, x^{'} \leq 1,$$

where $H^{'}$ is the derivative of H. For $\alpha = 1$ $(\alpha = 2)$membership in $HOLDER^{\alpha}$ imposes a Lipschitz condition on H (respectively on $H^{'}$); for $0 < \alpha \leq 1$ we are measuring a degree of fractional regularity of H, and for $1 < \alpha \leq 2$ a degree of fractional regularity of $H^{'}$. We define a functional class $HORIZ^{\alpha}(C_1, C_\alpha)$,

$$HORIZ^{\alpha}(C_1, C_\alpha) = \{f : H \in HOLDER^{\alpha}(C_\alpha) \cap HOLDER^{1}(C_1)\}. \tag{6.1}$$

This model is essentially the model of boundary fragments. Recall from Chapter 2, we have that the normal multiresolution approximation and thus normal polylines are closely related to certain well known fractal curves.

6.2 Interpolation on Fractal Curves

A special class of closed subsets Ω of $\mathbb{R}^n$, referred to as sets preserving Markov's inequality, are considered. Typically, Ω may be a fractal such as the Cantor set or von Koch's curve, but F may also be a closed Lipschitz domain. We investigate interpolation to smooth functions on Ω where the points of interpolation on $\partial\Omega$.

It should be noted that even if the sets which we primarily have in mind in this dissertation are fractals preserving Markov's inequality, the results are true for any sets F preserving Markov's inequality. The point we want to stress, however, is that the methods are applicable not only to certain sets with a nice geometry, like a Lipschitz domain or $\mathbb{R}^n$ itself, but also to certain very irregular sets.

Let $F \subset \mathbb{R}^n$ be a closed, nonempty subset (usually preserving Markov's inequality) and $B(x, r)$ is the closed n–dimensional ball with center $x \in F$ and radius $r \leq 1$. μ is a positive nontrivial Borel measure finite on bounded sets with support of μ. Π_k is the set of all polynomials in n real variables of total degree at most k.

Recall the definition of the d–dimensional Hausdorff measure of E, $m_d(E)$, where d is a positive number and $E \subset \mathbb{R}^n$.

$$m_d(E) = \lim_{\epsilon \downarrow 0} m_d^{\epsilon}(E),$$

where, for some positive constant $\alpha(d)$,

$$m_d^{\epsilon}(E) = \alpha(d) \inf \Big\{ \Sigma_j (diam E_j)^d : \bigcup_{i}^{\infty} E_j \supset E_j, diam E_j \leq \epsilon \Big\}.$$

Here $diamE_j$ is the diameter of E_j. We choose $\alpha(d)$ so that $m_d(E)$ coincides with the n–dimensional outer Lebesgue measure of E. The d–dimensional Hausdorff measure is an outer metric measure and the class of sets measurable $m_d(E)$ contains the Borel sets in $\mathbb{R}^n$. The Hausdorff dimension of E, $dim(E)$, is the infinimum of the set of $d > 0$ such that$m_d(E) = 0$. It is easy to see that $0 \le dim(E) \le n$.

Next, we define the concepts of d–measure and d–set. The set F is a d–set $(0 < d \le n)$ if there exists a μ with supp $\mu = F$ such that, for some positive constants $c_l = c_1(F)$ and $c_2 = c_2(F)$,

$$c_1 r^d \le \mu(B(x,r)) \le c_2 r^d \quad for\ x \in F\ and\ 0 < r \le 1.$$

Such a μ is called a d–measure on F. If F is a d–set then $m_d|F$, the restriction to F of the d–dimensional Hausdorff measure, is a d–measure on F. Also, any d–measure μ on F is equivalent to $m_d|F$ in the sense that, for some constants d_1 and d_2, $d_1\mu \le m_d|F \le d_2\mu$. Finally, if F is a d–set, then $dim(F) = d$ and $dim(F \cap B(x,r)) = d$ for all $x \in F$ and $r > 0$. $\mathbb{R}^n$ itself and a closed domain in $\mathbb{R}^n$ with boundary locally in $Lip_M 1$ are examples of d–sets with $d = n$ where the Lebesgue measure gives the d–measure.

Definition 6.2.1 *Let $F \subset \mathbb{R}^n$ preserves Markov's inequality if for every positive integer k there exists a constant $c = c(n,k,F)$ such that, for all polynomials $P \in \Pi_k$ and all balls $B = B(x_0, r)$, $x_0 \in F$, $0 < r \le 1$, we have*

$$\max_{F\cap B} |\nabla P| \le \frac{c}{r} \max_{F\cap B} |P|. \tag{6.2}$$

We refer (6.2) as Markov's inequality on F and note that for $F = \mathbb{R}^n$ it is the ordinary Markov inequality in $\mathbb{R}^n$. We are not concerned here with finding the best or even a good constant c but are just interested in using (6.2) as a condition on F. The following properties hold.

P1. F preserves Markov's inequality if and only if

$$\max_{B} |P| \le c \max_{F\cap B} |P|, \quad c = c(n,k,F),$$

for all k, all $P \in \Pi_k$, and all $B = B(x_0, r)$, $x_0 \in F$, $0 < r \le 1$.

P2. **Geometric Characterization**. F preserves Markov's inequality if and only if, for some $\varepsilon > 0$, none of the sets $B \cap F$, where $B = B(x_0, r)$, $x_0 \in F$, $0 < r \leq 1$, is contained in any band of type

$$\Big\{x \in \mathbb{R}^n : |b \cdot (x - x_0)| < \varepsilon r\Big\},$$

where $b \in \mathbb{R}^n$ and $|b| = 1$.

This property means that F is locally never too fiat but that F in a way extends in all n dimensions. This property is used in constructing basis function using Hardy's multiquadric in n dimensions.

P3. If $F \subset \mathbb{R}^n$ is a d–set with $d > n - 1$, then F preserves Markov's inequality. This property gives us that F preserving Markov's inequality, closure of a domain with boundary locally in $Lip_M 1$.

P4. F preserves Markov's inequality if and only if there exists a constant $d > 0$ so that, for every $B = B(x_0, r)$ where $x_0 \in F$ and $0 < r \leq 1$, there are $n + l$ affinely independent points $a_i \in F \cap B$, $i = 1, \cdots, n + 1$ such that the n–dimensional ball inscribed in the convex,hull of $a_1, \cdots a_{n+1}$ has radius not less than d.

We assume that $X = \{a_1, \cdots, a_{n+1}\} \subset \partial\Omega$, where $a_1, \cdots, a_{n+1}$ are, for instance, chosen in accordance with P4.

6.2.1 Besov Spaces on a d–set

We start by introducing some more notions related to the smoothness of a domain in $\mathbb{R}^n$. Let $\mathcal{H}^d$ be the d-dimensional Hausdorff measure. A Borel set $\Gamma \in \mathbb{R}^n$ is called a d-set, $0 < d \leq n$, if there exist positive constants c_1 and c_2 such that

$$c_1 r^d \leq \mathcal{H}^d(\Gamma \cap B(x, r)) \leq c_2 r^d \quad for \quad x \in \Gamma, \quad 0 < r \leq 1. \tag{6.3}$$

The notion d-set occurs both in the theory of function spaces and in fractal geometry. Clearly $\mathbb{R}^n$ is a d-set with $d = n$ and any convex compact set in $\mathbb{R}^n$ with non-empty interior is a d-set with $d = n$. Geometrically self-similar sets are typical examples of d-sets. In particular the Cantor set in $\mathbb{R}$ and von Kochs snowflake curve in $\mathbb{R}^2$ are d-sets with $d = \frac{\log 2}{\log 3}$ and $d = \frac{\log 4}{\log 3}$, respectively.

We say that the boundary $\partial\Omega$ of an open subset Ω of $\mathbb{R}^n$ is minimally smooth if there exist an $r > 0$, an integer N, a number $M > 0$ and a sequence (finite or infinite) $U_1, U_2, \cdots$ of open sets such that:

1. if $x \in \partial\Omega$ then $B(x, r) \subset U_i$ for some i;
2. no point of $\mathbb{R}^n$ is contained in more than N of the sets U_i ;
3. for each i there exists an G_i with $U_i \cap \Omega = U_i \cap G_i$ where G_i is the rotation of a Lipschitz domain of points in $\mathbb{R}^n$ of the form
$$\{x = (x', t) : t > \Phi(x'), x' \in \mathbb{R}^{n-1}, t \in \mathbb{R}\},$$
where $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ is a function (which may depend on G_i) satisfying a Hölder–Lipschitz condition with bound M.

Remark 6.2.1 *If Ω is an open subset of $\mathbb{R}^n$ having Lipschitz boundary $\partial\Omega$ then $\partial\Omega$ is a d–set with $d = n - 1$ and the closure of Ω is a d–set with $d = n$.*

Besov spaces on d–sets can be defined in terms of atomic decompositions. Let $\mathcal{N}_j$ be the set of half–open dyadic cubes of order j; for
$$Q = \left[\frac{k_1}{2^j}, \frac{k_1+1}{2^j}\right) \times \cdots \times \left[\frac{k_n}{2^j}, \frac{k_n+1}{2^j}\right) \in \mathcal{N}_j, \quad let\ \eta_Q = \left(\frac{k_1}{2^j}, \cdots, \frac{k_n}{2^j}\right).$$
For a dyadic cube Q of order j, $\tilde{Q}$ is the cube with the same center as Q and side length equal to $\frac{3}{2^j}$. We use standard multi–index notation, in particular $l = (l_1, l_2, \cdots, l_n)$ denotes a multi–index with length $|\ell| = \ell_1 + \ell_2 + \cdots + \ell_n$. Now, following Jonsson and Kamont (2001), we define less smooth atoms as:

Definition 6.2.2 *Let $1 \le p \le \infty$, $0 < \alpha$, $K = [\alpha]$, and let $\alpha < \beta \le K + 1$. Let Q be a dyadic cube of order j. A function $a : \mathbb{R}^n \to \mathbb{R}$ is called an (α, β, p)–atom corresponding to Q if and only if $a \in C^K(\mathbb{R}^n)$ and*
$$supp\ a \subset \tilde{Q}, \tag{6.4}$$

$$|D^\ell a(x)| \leq 2^{\ell(d/p+|\ell|-\alpha)} \quad for\ x \in \mathbb{R}^n,\ |\ell| \leq K, \tag{6.5}$$

$$|D^\ell a(x) - D^\ell a(y)| \leq 2^{\ell(d/p+|\ell|-\alpha)}|x-y|^{\beta-|\ell|} \quad for\ x, y \in \mathbb{R}^n,\ |\ell| = K, \tag{6.6}$$

Now, define Besov spaces $B^\alpha_{p,q}(F)$ with atomic decomposition as:

Definition 6.2.3 *Let $1 \leq p, q \leq \infty$, $\alpha > 0$, $K = [\alpha]$ and let $\alpha < \beta \leq K+1$. A function $f :\to \mathbb{R}$ belongs to $B^\alpha_{p,q}(F)$ if and only if there are sequences $\{a_Q : j \geq 0, Q \in \mathcal{N}_j\}$ of (α, β, p)–atoms and $\{v_Q : j \geq 0, Q \in \mathcal{N}_j\}$ of real coefficients, such that*

$$f = \sum_{j=0}^{\infty} \sum_{Q \in \mathcal{N}_j} v_Q a_Q, \tag{6.7}$$

with the series convergent in $L_p(F)$, and

$$\left(\sum_{j=0}^{\infty} \left(\sum_{Q \in \mathcal{N}_j} |v_Q|^p \right)^{q/p} \right)^{1/q} < \infty, \tag{6.8}$$

with the sums replaced by the respective suprema in case $p = \infty$ or $q = \infty$. The norm of f is defined as

$$\|f\|_{B^\alpha_{p,q}(F)} = \inf \left\{ \left(\sum_{j=0}^{\infty} \left(\sum_{Q \in \mathcal{N}_j} |v_Q|^p \right)^{q/p} \right)^{1/q} \right\}, \tag{6.9}$$

with the infimum taken with respect to all atomic decompositions of f .

We remark that in the above definition, condition $j \geq 0$ can be replaced by $j \geq j_0$ with any fixed $j_0 \in \mathbb{Z}$ and recall that in case $p = q = \infty$ and $0 < \alpha < 1$, these Besov spaces coincide with the usual spaces $Lip(\alpha, F)$ of functions satisfying the Hölder condition on F with the exponent α.

6.3 Characterization of Besov spaces on Minimally Smooth Boundary

Despite the elementary nature of the interpolating transform, it yields reasonably strong characterization for Hölder classes C^α, $0 < \alpha < 1$ and for the Besov classes embedding into some C^α. We develop results for Besov classes. This implies immediately results for Hölder (–Zygmund) and L_2–Sobolev classes.

Let $\partial\Omega$ be the minimally Lipschitz boundary of bounded domain Ω of $\mathbb{R}^n$ then we have $\partial\Omega$ be a d–set with d-measure μ, let $\{T_i\}_{i\geq 0}$ be a admissible sequence of triangulations of $\partial\Omega$, with normal multiresolution approximation. Then we assume that $\{\psi_\xi\}_{\xi\in\mathcal{V}}$ be the corresponding interpolating Schauder type basis, described in Proposition 5.5.1.

Remark 6.3.1 *Note that if $\alpha > d/p$ then $B^\alpha_{p,q}(\overline{\Omega}) \subset C(\overline{\Omega})$, which is a consequence of the trace theorems for Besov spaces on $\overline{\Omega}$ and on $\mathbb{R}^n$ and of the embedding theorems for Besov spaces on $\mathbb{R}^n$. If $f \in B^\alpha_{p,q}(\overline{\Omega})$ with $\alpha > d/p$, then the reconstruction of f from its samples $c_{j,\xi}(f)$ converges to f in the $B^\alpha_{p,q}$ norm with appropriate ordering of individual terms in topographic order is an important fact.*

The proof of this remark: simply that the norm of the sequence consisting of the coefficients in which were omitted in forming the approximation. Thus, if $f \in C^\alpha$, $0 < \alpha < 1$, use interpolating basis function of Hölder's regularity, then finite interpolating expansion converges in C^α norm. ∎

The following theorem is derived from Theorem 5.1 in Jonsson and Kamont (2001) in view of admissible sequence of functions.

Theorem 6.3.1 *Let $f \in C(\overline{\Omega})$,*

$$f = \sum_{j=0}^{\infty}\sum_{\xi\in\mathcal{V}_j} c_\xi(f)\psi_{j,\xi},$$

where the coefficients $c_\xi(f)$ are given by (5.25). Let $1 \leq p, q \leq \infty$ and $d/p < \alpha < 1$. Then, $f \in B^\alpha_{p,q}(\overline{\Omega})$ if and only if

$$C^\alpha_{p,q}(f) := \left(\sum_{j=0}^{\infty}\left(2^{js(d/p-\alpha)}\left(\sum_{\xi\in\mathcal{V}_j}|c_\xi(f)|^p\right)^{1/p}\right)^q\right)^{1/q} < \infty, \quad s \in \mathbb{R}, \tag{6.10}$$

with the sums replaced by the the respective suprema in case $p = \infty$ or $q = \infty$. Further, $C^\alpha_{p,q}$ is an equivalent norm in $B^\alpha_{p,q}(\overline{\Omega})$.

Proof : The proof follows by similar lines as proof in Jonsson and Kamont (2001) (Theorem 5.1) except some technical modifications, for example see Triebel (1999). The proof below is done in case $p < \infty$ and $q < \infty$, but the cases $p = \infty$ or $q = \infty$ require just technical modifications.

For $j \in \mathbb{Z}$, denote

$$I(j) = \left\{ i \geq 0 : r\delta_i \sim 2^{-js} \right\}, \tag{6.11}$$

where r is taken from Proposition 5.4.1, and let

$$j_0 = \min \left\{ j \in \mathbb{Z} : I(j) \neq 0 \right\}. \tag{6.12}$$

Note that condition T2 guarantees that there is $m \in \mathbb{N}$ such that for all j

$$\# I(j) \leq m. \tag{6.13}$$

At first, assume that $f \in C(F)$ and (6.10) holds. We construct a suitable atomic decomposition of f. For $\xi \in \mathcal{V}$, let $\tilde{\psi}_{j,\xi}$ be the extension of $\psi_{j,\xi}$ to $\mathbb{R}^n$, given by Proposition 5.4.1. For a dyadic cube Q of order j (half-open), let

$$\mathcal{V}(Q) = \left\{ \xi \in \mathcal{V} : \xi \in \mathcal{V}_j \cap Q \ for\ some\ \xi \in I(j) \right\},$$

$$c_Q = \left\{ |c_Q| : \xi \in \mathcal{V}(Q) \right\},$$

$$a_Q = \frac{2^{js(d/p-\alpha)}}{mc_Q} \sum_{\xi \in \mathcal{V}_j} c_\xi(f) \tilde{\psi}_{j,\xi}.$$

Then, $supp\ a_Q \subset \tilde{Q}$ and

$$\|a_Q\|_\infty \leq 2^{js(d/p-\alpha)},$$

since $\#\mathcal{V}(Q) \leq C$ (cf 6.13 and Proposition 5.4.1), we also have

$$|a_Q(x) - a_Q(y)| \leq 2^{js(1+d/p-\alpha)} |x - y|,$$

with C_1 depending only on $c7$ and r from Proposition 5.4.1, so a_Q/C_1 is an $(\alpha, 1, p)$–atom corresponding to Q. Since $\mathcal{V}(Q1) \cap \mathcal{V}(Q2) = \emptyset$ for $Q1 \neq Q2$, we have

$$f = \sum_{j=j_0}^{\infty} \sum_{Q \in \mathcal{N}_j} \frac{mc_Q}{2^{js(d/p-\alpha)}} a_Q,$$

with the series convergent uniformly on $\partial\Omega$; the definition of $I(j)$, (6.13) and (6.10) imply now

$$\left(\sum_{j=j_0}^{\infty}\left(\sum_{Q\in\mathcal{N}_j}\left(\frac{mc_Q}{2^{js(d/p-\alpha)}}\right)^p\right)^{q/p}\right)^{1/q} \leq CC^{\alpha}_{p,q}(f)<\infty,$$

i.e., this is the required atomic decomposition of f , and moreover there is C_2 such that

$$\|f\|_{B^{\alpha}_{p,q}(\overline{\Omega})}\leq C_2C^{\alpha}_{p,q}(f).$$

To prove the converse, let $f\in B^{\alpha}_{p,q}(\overline{\Omega})$ and let

$$f=\sum_{j=0}^{\infty}\sum_{Q\in\mathcal{N}_j}v_Qa_Q$$

be an atomic decomposition of f into $(\alpha,1,p)$–atoms such that

$$\left(\sum_{j=0}^{\infty}\left(\sum_{Q\in\mathcal{N}_j}|v_Q|^p\right)^{q/p}\right)^{1/q} \leq 2\|f\|_{B^{\alpha}_{p,q}(\overline{\Omega})}.$$

Since $\alpha>d/p$ and

$$\max_{x\in\overline{\Omega}}|\sum_{Q\in\mathcal{N}_j}v_Qa_Q(x)|\leq C2^{d/p-\alpha}\sup_{Q\in\mathcal{N}_j}|v_Q|\leq C2^{d/p-\alpha}\left(\sum_{Q\in\mathcal{N}_j}|v_Q|^p\right)^{1/p},$$

the series $\sum_{j=0}^{\infty}\sum_{Q\in\mathcal{N}_j}v_Qa_Q$ converges uniformly on $\overline{\Omega}$, and consequently

$$c_\xi(f)=\sum_{j=0}^{\infty}\sum_{Q\in\mathcal{N}_j}v_Qc_\xi(a_Q).$$

Formula (5.25) and Definition 6.2.2 of an $(\alpha,1,p)$–atoms we have

$$|c_\xi(a_Q)|\leq 2.2^{js(d/p-\alpha)} \tag{6.14}$$

and for $i>0$,

$$|c_\xi(a_Q)|\leq \delta_i 2^{js(1+d/p-\alpha)}\quad for\quad \xi\in\mathcal{V}_i. \tag{6.15}$$

For $i\geq 0$, let

$$j_i=\max\left\{j\geq 0:\delta_i\leq 2^{-js}\right\},\qquad j_i=0\ if\ \delta_i>1$$

and for $\xi\in\mathcal{V}_j$, let

$$x_\xi(f)=\sum_{0\leq j\leq j_i}\sum_{Q\in\mathcal{N}_j}v_Qc_\xi(a_Q),\quad y_\xi(f)=\sum_{j\geq j_i}\sum_{Q\in\mathcal{N}_j}v_Qc_\xi(a_Q).$$

Since supp $a_Q \in Q$, formula (5.25) implies the existence of a constant k such that for all ξ and j

$$\#\{Q \in \mathcal{N}_j : c_\xi(a_Q) \neq 0\} \leq k. \tag{6.16}$$

At first, we estimate

$$X_{i,p} = \left(\sum_{\xi \in \mathcal{V}_i} |x_\xi(f)|^p \right)^p .$$

Recall that $\partial\Omega$ is a d–set, and that for each $\eta, \varsigma \in \mathcal{U}_i$ holds $|\eta - \varsigma| \geq c\delta_i$ for each $Q \in \mathcal{N}_j$ with $j \leq j_i$, we have

$$\#\{\xi \in \xi \in \mathcal{V}_i : c_\xi(a_Q) \neq 0\} \leq C 2^{-dj} \delta_i^{-d}.$$

Using this, (6.15) and (6.15), we get by Minkowskis and Jensens inequalities we have

$$X_{i,p} \leq \sum_{0 \leq j \leq j_i} \left(\sum_{\xi \in \mathcal{V}_i} \left(\sum_{Q \in \mathcal{N}_j} |v_Q||c_\xi(a_Q)| \right)^p \right)^{1/p} \leq$$

$$\leq C \sum_{0 \leq j \leq j_i} \left(\sum_{\xi \in \mathcal{V}_i} \sum_{Q \in \mathcal{N}_j} |v_Q|^p |c_\xi(a_Q)|^p \right)^{1/p} \leq$$

$$\leq C \sum_{0 \leq j \leq j_i} \delta_i^{1-d/p} 2^{js(1-\alpha)} \left(\sum_{Q \in \mathcal{N}_j} |v_Q|^p \right)^{1/p} ,$$

which implies

$$X_{i,p,\alpha} = \delta_i^{d/p-\alpha} X_{i,p} \leq C \delta_i^{1-d/p} \sum_{0 \leq j \leq j_i} 2^{js(1-\alpha)} \left(\sum_{Q \in \mathcal{N}_j} |v_Q|^p \right)^{1/p} .$$

The last inequality, the choice of j_i (i.e. $\delta_i \tilde{2}^{-j_i s}$) and Jensens inequality imply

$$X_{i,p,\alpha} \leq C \delta_i^{1-d/p} \sum_{0 \leq j \leq j_i} 2^{js(1-\alpha)} \left(\sum_{Q \in \mathcal{N}_j} |v_Q|^p \right)^{1/p} . \tag{6.17}$$

Since by T2 there is an m' such that, for all ℓ, $\#\{i : j_i = \ell\} = m'$, we have

$$\sum_{i : j_i \geq j} \delta_i^{1-d/p} = \sum_{\ell=j}^{\infty} \sum_{i : j_i \geq j} \delta_i^{1-d/p} \leq C \sum_{\ell=j}^{\infty} 2^{-\ell(1-\alpha)} \leq C 2^{-js(1-\alpha)}.$$

Thus by (6.17)we get

$$\sum_{i=0}^{\infty} X_{i,p,\alpha}^q \leq C^q \sum_{j=0}^{\infty} \left(\sum_{Q \in \mathcal{N}_j} |v_Q|^p \right)^{q/p} . \tag{6.18}$$

By similar arguments we can estimate

$$Y_{i,p} = \left(\sum_{\xi \in \mathcal{V}_i} |x_\xi(f)|^p \right)^p .$$

There is a constant k_1 such that for $Q \in \mathcal{N}_j$ with $j > j_i$ such that

$$\#\{\xi \in \mathcal{V}_i : c_\xi(a_Q) \neq 0\} \leq k_1$$

Using this, Jensens inequality and inequalities (6.14) and (6.16), we get

$$Y_{i,p} \leq \sum_{j>j_i} \left(\sum_{\xi \in \mathcal{V}_i} \left(\sum_{Q \in \mathcal{N}_j} |v_Q||c_\xi(a_Q)| \right)^p \right)^{1/p} \leq$$

$$\leq C \sum_{j>j_i} \left(\sum_{\xi \in \mathcal{V}_i} \sum_{Q \in \mathcal{N}_j} |v_Q|^p |c_\xi(a_Q)|^p \right)^{1/p} \leq$$

$$\leq C \sum_{j>j_i} \delta_i^{d/p-\alpha} 2^{js(1-\alpha)} \left(\sum_{Q \in \mathcal{N}_j} |v_Q|^p \right)^{1/p} ,$$

which implies

$$Y_{i,p,\alpha} = \delta_i^{d/p-\alpha} Y_{i,p} \leq C \delta_i^{d/p-\alpha} \sum_{j>j_i} 2^{js(d/p-\alpha)} \left(\sum_{Q \in \mathcal{N}_j} |v_Q|^p \right)^{1/p} .$$

Since $\alpha > d/p$ and $\delta_i \sim 2^{-j_i}$

$$Y_{i,p,\alpha} \leq C \delta_i^{d/p-\alpha} \sum_{j>j_i} 2^{js(d/p-\alpha)} \left(\sum_{Q \in \mathcal{N}_j} |v_Q|^p \right)^{1/p} .$$

Finally we get

$$\sum_{i=0}^{\infty} Y_{i,p,\alpha}^q \leq C^q \sum_{j=0}^{\infty} \left(\sum_{Q \in \mathcal{N}_j} |v_Q|^p \right)^{q/p} .$$

Since $c_\xi(f) = x_\xi(f) + y_\xi(f)$, the last inequality and (6.18) imply (6.10). Hence the result. ∎

For the later convenience, let $\mathcal{V} = \cup_{j=0}^{\infty} \mathcal{V}_j$, and let $\preccurlyeq$ be a linear order on $\mathcal{V}$ satisfying the following conditions:

$$\textit{if } \xi \in \mathcal{V}_i \textit{ and } \eta \in \mathcal{V}_j \textit{ with } i < j, \textit{ then } \xi \preccurlyeq \eta. \tag{6.19}$$

6.3.1 Orthonormal Franklin Basis in $L_2(\overline{\Omega})$

Let $\partial\Omega$ be a d–set, and let $\{T_i\}_{i\geq 0}$ be a admissible sequence of triangulations of $\partial\Omega$. Let μ be the d–measure on $\overline{\Omega}$. By the orthonormalization of the functions $\{\psi_\xi, \xi \in \mathcal{V}\}$ in $L_2(\overline{\Omega})$, we obtain an orthonormal system $\{f_\xi, \xi \in \mathcal{V}\}$ of continuous piecewise linear functions. More precisely, let $\preccurlyeq$ be a linear order on $\mathcal{V}$ satisfying (4.2). For each $\xi \in \mathcal{V}$, there is a unique (up to a sign) function $f.$ such that $f_\xi \in span\{\phi_\eta, \eta \preccurlyeq \xi\}$, f_ξ is orthogonal in $L_2(\overline{\Omega})$ to $span\{\phi_\eta, \eta \preccurlyeq \xi\}$ and $\|f\|_{L_2} = 1$. The system $\{f_\xi, \xi \in \mathcal{V}\}$ is an analogue on $\overline{\Omega}$ of the classical Franklin system.

For $\xi \in \mathcal{V}_j$, $j \geq 1$, let the coefficients $\{c_{\xi,\eta}\}_{\eta\in\mathcal{U}_j}$ be such that

$$f_\xi = \sum_{\eta\in\mathcal{U}_j} c_{\xi,\eta}\phi^\star_{j,\eta}. \tag{6.20}$$

Moreover, let

$$P_\xi f = \sum_{\eta\preccurlyeq\xi} (f, f_\eta) f_\eta$$

be the orthogonal in $L_2(\overline{\Omega})$, projection onto $span\{f_\eta : \eta \preccurlyeq \xi\}$. It follows from Proposition 5.5.1 and the density of continuous functions in $L_p(\overline{\Omega})$ for $1 \leq p < \infty$, that the collection of functions $\{f_\xi : \xi \in \mathcal{V}\}$ is linearly dense in the spaces $C(\overline{\Omega})$ and $L_p(\overline{\Omega})$ for $1 \leq p < \infty$. Further, it is proved in (Jonsson and Kamont, 2001) there is a finite constant C such that for all $\xi \in \mathcal{V}$ and $1 \leq p \leq \infty$

$$\|P_\xi\|_p = \|P_\xi : L_p(\overline{\Omega}) \to L_p(\overline{\Omega})\| \leq C.$$

Moreover, the set of functions $\{f_\xi\}_{\xi\in\mathcal{V}}$ (ordered with respect to $\preccurlyeq$ satisfying (6.19)) is a basis in $L_p(\overline{\Omega})$ for $1 \leq p < \infty$ and in $C(\overline{\Omega})$. Then, the Theorem 6.1 stated in (Jonsson and Kamont, 2001) modified as:

Theorem 6.3.2 *Let $1 \leq p \leq \infty$ and $0 < \alpha < 1$. Let $f \in L_p(\overline{\Omega})$ for $1 \leq p < \infty$ and $f \in C(\overline{\Omega})$ in case $p = \infty$,*

$$f = \sum_{i=0}^{\infty} \sum_{\xi\in\mathcal{V}_i} d_\xi \psi_\xi, \quad \textit{where } d_\xi = (f, f_\xi).$$

Then, $f \in B^{\alpha}_{p,q}(\overline{\Omega})$ *if and only if*

$$D^{\alpha}_{p,q}(f) := \left(\sum_{j=0}^{\infty} \left(2^{js(d/p-d/2-\alpha)} \left(\sum_{\xi\in\mathcal{V}_j} |d_\xi|^p \right)^{1/p} \right)^q \right)^{1/q} < \infty, \quad 0 < s < 1, \tag{6.21}$$

with the sums replaced by the the respective suprema in case $p = \infty$ *or* $q = \infty$*. Further,* $D^{\alpha}_{p,q}$ *is an equivalent norm in* $B^{\alpha}_{p,q}(\overline{\Omega})$*.*

Proof : The proof follows by similar lines of arguments as in Theorem 6.3.1(cf Theorem 6.1 in Jonsson and Kamont (2001)). ∎

Let us note that the Theorem 6.3.2 implies also the following: let p, q, and a be as in Theorem 6.3.2, and let $\{d_\xi : \xi \in \mathcal{V}\}$ be a sequence of real numbers satisfying (6.21). Then the series $\sum_{j=0}^{\infty} \sum_{\xi\in\mathcal{V}_j} d_\xi f_\xi$ converges in $L_p(\overline{\Omega})$ ($C(\overline{\Omega})$ in case $p = \infty$), and for

$$f = \sum_{j=0}^{\infty} \sum_{\xi\in\mathcal{V}_j} d_\xi f_\xi,$$

we have

$$d_\xi(f) = (f, f_\xi) = d_\xi \quad for\ f \in B^{\alpha}_{p,q}(\overline{\Omega}).$$

An analogous fact also follows from the proof of Theorem 6.3.1 for the system $\{\psi_\xi, \xi \in \mathcal{V}\}$ and sequences $\{c_\xi, \xi \in \mathcal{V}\}$ satisfying (6.10) (with convergence in $C(\overline{\Omega})$). Thus, we have

Corollary 6.3.1 *Let* $\partial\Omega$ *be a d–set, let* $\{T_j\}_{j\geq 0}$ *be a admissible sequence of triangulations of* $\partial\Omega$*, and let* $\mathcal{V}_i$*,* $\mathcal{V}$ *be the appropriate sets of vertices. For* $1 \leq p, q \leq \infty$ *and* $0 < \alpha < 1$*, let*

$$b^{\alpha}_{p,q} = \left\{ b = \{b_\xi\}_{\xi\in\mathcal{V}} : b^{\alpha}_{p,q} = \left(\sum_{j=0}^{\infty} \left(2^{js(d/p-d/2-\alpha)} \left(\sum_{\xi\in\mathcal{V}_j} |d_\xi|^p \right)^{1/p} \right)^q \right)^{1/q} < \infty \right\},$$

with the sums replaced by respective suprema in case $p = \infty$ *or* $q = \infty$*. Let* $\{f_\xi, \xi \in \mathcal{V}\}$ *be the orthonormal Franklin type basis. Then,*

$$f \mapsto \{(f, f_\xi)\}_{\xi\in\mathcal{V}}$$

is a linear isomorphism of $B^{\alpha}_{p,q}(\overline{\Omega})$ and $b^{\alpha}_{p,q}$. In case $\alpha > d/p$, another isomorphism of $B^{\alpha}_{p,q}(\overline{\Omega})$ and $b^{\alpha}_{p,q}$ is given by

$$f \mapsto \{2^{jsd/2} c_{\xi}(f)\}_{\xi \in \mathcal{V}},$$

where $c_{\xi}(f)$, given by formula (4.3), are the coefficients of f with respect to the interpolating Schauder type basis.

The next corollary is important.

Corollary 6.3.2 *Let $\partial\Omega$ be a d–set, let $\{T_j\}_{j\geq 0}$ be a admissible sequence of triangulations of $\partial\Omega$. Let $\{\psi_{\xi}, \xi \in \mathcal{V}\}$, $\{f_{\xi}, \xi \in \mathcal{V}\}$ be the Schauder and Franklin type bases, respectively, and let $\psi^{\star}_{\xi} = 2^{jsd/2}\psi_{\xi}$. Then, for each p, q, α with $1 \leq p \leq \infty$, $d/p < \alpha < 1$ and $1 \leq q < \infty$, the systems $\{\psi^{\star}_{\xi}, \xi \in \mathcal{V}\}$ and $\{f_{\xi}, \xi \in \mathcal{V}\}$ are equivalent bases in $B^{\alpha}_{p,q}(\overline{\Omega})$.*

Moreover, for $q = \infty$, $1 \leq p \leq \infty$, $d/p < \alpha < 1$ and any sequence of real coefficients $\{b_{\xi}, \xi \in \mathcal{V}\}$,

$$f = \sum_{i=0}^{\infty}\sum_{\xi\in\mathcal{V}_i} b_{\xi}\psi^{\star}_{\xi} \in B^{\alpha}_{p,\infty}(\overline{\Omega}) \quad ifandonlyif \quad g = \sum_{i=0}^{\infty}\sum_{\xi\in\mathcal{V}_i} b_{\xi} f_{\xi} \in B^{\alpha}_{p,\infty}(\overline{\Omega}),$$

and then $\|f\|_{B^{\alpha}_{p,\infty}(\overline{\Omega})} \sim \|g\|_{B^{\alpha}_{p,\infty}(\overline{\Omega})}$.

For $1 \leq p, q \leq \infty$ and $0 < \alpha < 1$, let with the sums replaced by respective suprema in case of $p = \infty$. Where $\alpha > d/p$ is essential rather then technical. The critical case $\alpha = d/p$ and $1 \leq p, q \leq \infty$.

6.3.2 Bounded Variation

It is much more obvious that the normal wavelet decomposition represent the space $BV(\Omega)$ in terms of some reasonable assumption and it is appropriate space represent gray–scale images. But the space BV does not have simple representation or even have a basis. This make the representation is complicated.

The space $BV := BV(\Omega)$ of functions of bounded variation on a domain $\Omega \subset \mathbb{R}^d$ is important in Mathematics (geometric measure theory, differential geometry) and applications (image processing, nonlinear PDEs). The structure of BV is complicated by the fact that neither it nor the closely related Sobolev space $W^1(L_1(\Omega))$ have an unconditional basis. Wavelet decompositions of BV functions, while not characterizing this space, give fine information (Cohen et al., 2003, 1999) about its structure and these decompositions can be used to solve various extremal problems. Consider, for example, the extremal problem

$$K(f,t) := K(f,t; L_2(\Omega), BV(\Omega)) := \inf_{g \in BV(\Omega)} \|f-g\|_{L_2(\Omega)} + t|g|_{BV(\Omega)}, \tag{6.22}$$

where $t > 0$ is a parameter. The expression (6.22) is called a K–functional in interpolation of linear operators. It is used to describe interpolation spaces between $L_2(\Omega)$ and $BV(\Omega)$. This and related functionals also occur in image processing in such problems as denoising and deblurring. The rate of decay of $K(f,t)$ as $t > 0$ gives information about the smoothness of f relative to $L_2(\Omega)$ and $BV(\Omega)$.

Hence, we are looking for space which is closer to $BV(\Omega)$. The fact we could use here is that $B^1_{1,1} \subset BV \subset B^1_{1,\infty}$, i.e., $B^1_{1,1}$ is closer to BV. Then we would be able describe the information of BV in terms $B^1_{1,1}$. Hence, consider variation spaces V_p of Peetre (1976) instead the ℓ^p. These have seminorm

$$\|f\|_{V_p} = \sup_{t_i < t_{i+1}} \|(f(t_{i+1}) - f(t_i))_i\|_p,$$

where sup is all over the partitions of the line. Whence $p = 1$ is just the bounded variation seminorm. Peetre points out that

$$B^{d/p}_{p,1} \subset V_p \subset B^{d/p}_{p,\infty}.$$

Then we have

$$B^1_{1,1} \subset BV.$$

Then the space BV has been characterized by the normal wavelet coefficients as

$$\|\theta\|_{b^1_{1,1}} \geq C\|f\|_{BV},$$

where C is some positive constant.

6.4 Summary

The main argument point of Horizon class images, i.e., gray–scale images, are decomposed into less smooth function spaces with normal wavelets. This decomposition leads to nonlinear approximation of functions. In the next chapter nonlinear approximation of wavelet coefficients and related image compression are briefly discussed.

Chapter 7

Applications

7.1 Image Compression

This chapter concerns with approximation of functions which are represented by some kind of data. By approximation, it means approximating the functions in certain function spaces with a finite dimensional representation. Immediate applications of approximation consist compression and noise removal. For images and many other kinds of data, an approximation is typically defined on a discrete set of points on some grid. For example, digital images are typically acquired by sampling the light intensity at discrete points on a square grid of pixels (currently using a CCD array), and so image representations and processing algorithms typically operate on this square grid. The square pixel grid is nearly always assumed to be fixed with the dependent variable of the image, the pixel intensity. While the acquisition and processing of image data on a square grid of pixels is simple, it turns out to be very inefficient for representing many important image features including the edges.

A discrete triangle $\triangle$ is defined as the set of pixel locations (r, s) with non–overlapping but, adjacent rectangular areas $\square_{r,s}$ forming an area which is bounded by the area of three discrete edges having pairwise end–point in common. Using these definitions we can analogously define the discrete counterparts of a triangulation and mesh for which we use the notations $M := (V, E, F)$ and T. For the sake of brevity we will drop the word discrete in the remainder of this chapter.

Chapter 3 proposed a normal offset based approximation scheme generating piecewise linear approximants that interpolate a given function at the knots defined by the locations of the piercing points. In regions where the function is smooth, the normal mesh representation converges rapidly to the function graph. In the vicinity of a contour, however, extra vertical offsets need to be introduced to approximate the function graph, although the triangulation captures the geometry quite well.

This chapter built on the idea developed in Chapter 3 to approximate two–dimensional piecewise smooth functions using Hardy's multiquadric. The presented method is a nonlinear piecewise polynomial approximation method over normal multilevel triangulations (NMTs). We show experimental results for digital image encoding and compare them with the JPEG2000 encoder and low–bit rate image coder, such as adaptive geometric piecewise polynomial approximation (Kazinnik et al., 2007).

7.1.1 Surface Compression

The idea followed in this dissertation is to treat images as special cases of 2 D surfaces and represent them using triangular functions. Because, the triangles edges can be placed on arbitrary locations and orientations. Triangles have the potential to represent arbitrary edge contours (the geometry information) more accurately with a fewer number of patches than a fixed square grid representation. The key idea is to use an admissible triangulation that places vertices more densely in edge regions for accurate and efficient edge representation, yielding a parsimonious image representation.

Multiresolution triangulations meshes are widely used in computer graphics for representing 2 D surfaces and 1 D piecewise smooth functions, because triangles have potential to more efficient approximation at the discontinuities between the smooth pieces than the other standard tools like wavelets. The normal multiresolution mesh decomposition is an anisotropic representation of the 2 D surfaces. The same idea of anisotropic representation lies at the basis decomposition of wedgelets and curvelets transforms.

For efficient processing of 3 D mesh data, multiscale triangulation based on nonlinear subdivision has been proposed in computer graphics. Multiscale mesh construction starts from a small number of coarse-scale points on the surface. Finer triangular meshes are formed by subdividing, that is, by gradually adding more data points (vertices, pixels). Unlike the standard subdivision scheme that places new vertices at the midpoints of the triangle edges, we can adapt the location of the new vertices based on local geometry information. The normal mesh scheme selects the new points based on the local normal direction computed from the previous coarser scale mesh (Guskov et al., 2000). DeVore et al. (1992b) showed that compressed approximant s of $f \in B_q^\alpha(L_p(\Omega))$ with N terms satisfies

$$\|f - s\|_\infty \le C|f|_{B_q^\sigma(L_p(\Omega))} N^{-\sigma/2}, \quad \sigma < 3.$$

The following section gives a simple surface compression algorithm based on nonlinear thresholding of wavelet coefficients.

7.2 Compression of Quasi–Interpolating Expansions

It is shown that the norm of f in several smoothness classes can be determined from the size of the coefficients in the wavelet decomposition. In this section, we consider a simple surface compression algorithm and give an error bound for the approximation of f by its compressed wavelet decomposition. The most natural norm for compression is the L_∞ norm, hence our approximation results take place in this norm.

Suppose, we are given a function $f \in B_q^\alpha(L_p(\Omega))$, $\alpha > d/p$ that represents the surface, that is being compressed. The surface compression algorithm is as follows: let $j \ge j_0$, and $k \in K_j$, then we have a quasi–interpolating transform

$$f = \sum_{j \ge j_0} \sum_{k \in K_j} d_{j,k} \psi_{j,k},$$

for an interval with a very simple algorithmic structure. The interpolating transform for the domains gives a sparse representation of certain functions; particularly for piecewise polynomials. If f is a piecewise polynomial on Ω with less than or equal to P pieces, each of degree D, sampled on a dyadic grid of $n = 2^{j_1}$ points, then from those samples, we can calculate the wavelet coefficients at scale $j_1 - 1$ and each coarser scale, and there are only $C_0 + P \cdot (D+1) \cdot \log_2(n)$ nonzero wavelet coefficients among them.

General functions have many nonzero wavelet coefficients. But, often these can be well approximated by a sparse sequence. Suppose that we have the collection of all interpolating wavelet coefficients θ and, we sparsify these as follows: let $\epsilon > 0$ denote a thresholding control parameter and then

$$c^{\epsilon}_{j,\xi} = c_{j,\xi} \cdot 1_{\{|c_{j,\xi}| > \epsilon \cdot 2^{-jds/2}\}}, \quad for\ some\ s \in \mathbb{R}.$$

Hence, at each level $j \geq j_0$ we set to zero for coefficients which are smaller in amplitude than $2^{-jds/2}$.

Suppose that $f \in B^{\alpha}_{p,q}(\overline{\Omega})$, for some $\alpha > d/p$, then θ^{ϵ} has finitely many nonzero terms. The series obtained by summing the wavelet series corresponding to θ^{ϵ} produces a reconstruction f^{ϵ} : $f^{\epsilon} \to f$ in $B^{\alpha}_{p,q}(\overline{\Omega})$ as $\epsilon \to 0$. The sparse representation f^{ϵ} has advantages over another finite series. By Donoho (1992) (Theorem 3.8), if $\alpha > d/p$ then,for each $\eta > 0$ we can pick $\tilde{p} < p$ so that $\alpha - \eta < 1/\tilde{p} < \alpha$. Using $\|\theta\|_{b^{\alpha}_{p,\infty}} \leq \|\theta\|_{b^{\alpha}_{p,q}}$ we can define a sequence ϵ_n such that $N(\epsilon_n) \leq n$, hence,

$$\|f - f^{\epsilon_n}\| \leq C(p, q, \eta) \|f\|_{B^{\alpha}_{p,q}(\overline{\Omega})} n^{-(\alpha - \eta)}$$

is near optimal in a certain minimax compression model. Now Q_J has $n = 2^J$ nonzero terms, then

$$\|f - Q_J f\| \leq C \|f\|_{B^{\alpha}_{p,\infty}(\overline{\Omega})} n^{-(\alpha - d/p)},$$

except in the case $p = \infty$, this rate is slower than the rate $n^{-\alpha}$ nearly attained. Hence, Q_J has a slower minimax rate of approximation than f^{ϵ_n} in general; it gives worse reconstructions for a given number of nonzero terms.

7.3 Motivation

Edges are the dominating features in piecewise smooth 2 D surfaces. An edge contains two types of information where the edge is located, i.e., its location and the geometry. In 2 D, geometry information plays a crucial role, much more than in 1 D. In 1 D piecewise smooth functions, discontinuities occur at isolated points, and these can be easily captured in a wavelet transform. In 2 D, edge singularities lie along 1 D contours which are much harder to capture. The time–scale analysis of the wavelet representation provides a powerful tool for approximating a 1 D function f. Under mild conditions, a nonlinear wavelet approximation f_n containing the n largest terms of the wavelet expansion of f performs well on a certain function class such as Besov spaces (DeVore et al., 1992a,b). Indeed, the L_2 approximation error decays rapidly with increasing n:

$$\|f - f^{1-Dwavelet}\| = O(n^{-\nu}). \tag{7.1}$$

In this equation, ν stands for

$$\nu = \min(\tilde{p}, \alpha),$$

with $\tilde{p}$ the number of (dual) vanishing moments of the wavelet analysis and α the Lipschitz regularity of the signal at its non–singular points. Wavelets provide a very efficient representation of 1 D function with certain smoothness because, in a 1 D function, the geometry information consists of merely a few isolated points. Wavelets are thus well–suited for estimating a piecewise smooth 1 D function in presence of noise. In the minimax sense, the performance of a simple n–term approximation algorithm comes within a negligible logarithmic factor of the best possible method involving a piecewise polynomial with knots at the (assumed known) positions of the singularities (Donoho, 1994).

Unfortunately, this approximation does not carry over into two and higher dimensions. Indeed, standard tensor–product wavelet transforms based on a square grid of 2 D sampling points are ill–prepared to represent edges, since many wavelets overlap with 1 D edges leading to a preponderance of geometry information. Given a 2 D function f, that is smooth except for an edge singularity along a smooth (say C^2) curve, the nonlinear wavelet approximation f_n using the n largest wavelet terms has an L_2 error rate

$$\|f - f^{2-Dwavelet}\| = O(n^{-1/2}).$$

This outperforms a Fourier procedure, where the best we can do is a linear approximation, taking the first n Fourier coefficients

$$\|f - f^{2-DFourier}\| = O(n^{-1/4}).$$

Nevertheless, neither of these procedures comes close to the 1 D rate of (7.1). This is partly due to an inherent dimensionality effect: approximation of 2 D data is inevitably more difficult than 1 D data. Yet, wavelets do not obtain the optimal 2 D rate either. They approximate a curved singularity as a piecewise constant. This observation explains the blocky output of wavelet image approximations.

In order to achieve better approximation rates on 2 D edge contours, new edge–adaptive multiscale decompositions have been developed in recent years. Due to the numerous possible orientations, lengths and curvatures of edges, it is impossible to catch all possible edges by a basis decomposition. The new multiscale decomposition may proceed in over complete representations (frames), such as contourlets (Do and Vetterli, 2003) or curvelets (Candès and Donoho, 2000).

Although, interpolating normal approximations have an optimal n–terms convergence rate (for $n \to \infty$) for smooth curves (Daubechies et al., 2004; Guskov et al., 2000) and have been used in practice for approximating smooth 2–D manifolds (Guskov et al., 2000). The interpolating meshes or polylines will be highly aliased approximants for relatively few terms. Indeed, top–down construction of interpolating meshes works with unfiltered subsamples from the input data. For this reason Friedel et al. (2004) proposed to use approximating normal meshes. The application of normal meshes for real images is subject of current research. The results in this dissertation should therefore be seen as provisional. Normal meshes could be used for image modeling, compression, and processing. However, algorithms will have to take into account that the decomposition is highly nonlinear. Normal offsets are the key to adaptive triangulation of 2 D data sets. These data may contain line singularities and posing substantial problems to any tensor product based decomposition.

Aliasing effects are even more noticeable when the objects contain jumps. For the two-dimensional case, Jansen et al. (2005) give an asymptotic convergence analysis for the simple case of Horizon class images. However, the cited work does not give a treatment of more general objects that are piecewise smooth (instead of constant) objects added with textures and noise. For low bit rates, the number of segments in the final partition has to be limited, and an interpolating normal mesh will result in poor approximations of the function graph. Further implementation on natural scene images by Aerschot (2009) does not give satisfactory results, i.e., the compression ratio is approximately 1.1bpp. Moreover, its tree structured coefficient selection has to deal with the topological aspects. The nonlinear character of the decomposition itself makes it harder to analyze the effect of removing or modifying a given coefficient. In 2 D, the topological exceptions complicate the whole thing: changing a single coefficient may influence the topology on the following finer grids. But, it is difficult to deal with different types of images in a cohesive manner. Therefore, it is equipped with the interpolating normal meshes with Hardy's multiquadric basis functions.

7.4 Digital Images

The image, viewed as a function, is approximated by a linear spline over the normal multiresolution triangulation of a small adaptively chosen set of significant pixels, such pixels capture the geometry of the image. In general, the significant pixels are scattered in the rectangular image domain. Their normal multiresolution triangulation is anisotropic. All linear splines over this adaptive triangulation constitute a suitable approximation space for the image, from which we take the best approximation to the image by minimizing the mean square error. This linear spline is a continuous function which can be evaluated at any point in the rectangular image domain, in particular, at the discrete set of pixels. Indeed, the compressed image is reconstructed from this linear spline. Moreover, our specific representation of the image (by a continuous function) allows us to display the reconstructed image on any subset of the (continuous) image domain. This option is especially relevant for applications such as zooming, rescaling and conversion between different image representations.

For implementation of this scheme we need triangulation of the image. There are several types of triangulation, such as Delaunay triangulation, but for our purpose Normal Multiresolution Triangulation (NMT) is considered, since it shows the multiresolution properties. The adjustments to be made in order to implement a normal mesh based decomposition. For instance, obtaining nested triangulations on a regular grid is more cumbersome than in the continuous setting.

The content of $f \in \mathcal{H}$ is completely determined by the planar curve γ. The goal is to approximate γ using an adaptive triangulation method such that the triangle boundaries form a piecewise affine continuous (i.e. polyline) approximation of γ. This is in contrast with tensor product wavelets where γ is approximated by the borders of the rectangular supports creating a piecewise constant approximation of γ. Given a triangular mesh, $\mathcal{M}_0$ with vertices $\mathcal{V}_0$, edges $\mathcal{E}_0$ and triangles $\mathcal{T}_0$, the normal offset scheme consists of an iterative application of three steps.

1. The first step, the prediction step, constructs additional vertices – prediction points – as linear combinations of surrounding mesh points.
2. The second step, the correction step, constructs piercing points as the intersection of rays normal to the coarser mesh, going through those prediction points and the image surface.
3. The last step, the interconnection step, adds all piercing points to the set of vertices $\mathcal{V}_j$ forming $\mathcal{V}_{j+1}$. The corresponding mesh $\mathcal{M}_{j+1}$ with edges $\mathcal{E}_{j+1}$ and triangles $\mathcal{T}_{j+1}$ is constructed by a triangulation of $\mathcal{V}_{j+1}$.

These steps are repeated, creating meshes at different resolution levels j, until a certain stopping criterion is satisfied.

The created piercing points have their (x, y) coordinate on the projection of the triangle edge from which the normal ray was shot. After the vertex insertion step, each triangle is locally subdivided by creating edges connecting each piercing point to another piercing point or triangle vertex that belong to the same triangle. Hence, all edges in $\mathcal{E}_{j+1}$ are either obtained by an edge refinement of a coarser edge in $\mathcal{E}_j$, or inserted as a connection between admissible combinations of piercing point or triangle vertices.

As each triangle is split into four subtriangles. The nested triangulation can be represented as a quadtree. With each node of this quadtree we can connect a triangle subdivision operator. The collection of triangles at nodes having the same level form a conforming triangulation. A conforming triangulation requires that two triangles have either an edge in common or have no point in common. The triangle subdivision rule can be fixed in advance or can be made data–dependent.

When the splitting rule is fixed in advance, no extra topological information has to be stored, since the connection of new piercing points always happens in the same way. On the other hand the triangle splitting is blind with respect to the shape of the above lying surface connected as defined by f.

This already lets us assume that in the neighborhood of a contour, the discontinuity will not be approximated appropriately. The fact is piercing points always have correlation. In order to circumviate this problem adaptive triangulation is introduced by aligning with the contours of the image. But, here it is proposed to use Halton sequence of points described below. The piercing points are adjusted with Halton sequence of points as closest neighbour. Functions are sampled at Halton data sites generate set of points in the unit square $\Omega = [0, 1] \times [0, 1]$. For interpolation functions (images), the basis function $\phi(r) = (r^2 + c^2)^{1/2}$ is used as in (7.3), where $r = \|x - 2^{-j}k\|$, $k \in \mathbb{Z}$ and $c = 0.01$.

7.4.1 Geometric Images

The majority of images (rendered or captured) are projections of 3 D–scenes onto a lower dimensional (2 D) space. In general, a natural scene consists of a collection of smoothly shaded objects demarcated by smooth boundaries. In case of only one object is present, its boundary will contrast with the background illumination resulting in a smoothly varying contour representing the objects silhouette. As such, the contour holds a crucial part of the information contained in the image. This is acknowledged by the ability of the human visual system to interpret simple line drawings or sketches and connect them to real–world objects. In case of multiple objects, the image is a nonlinear superposition (caused by blending or occlusion) of the projections of each object separately. Moreover, contours may be interrupted at places where another object appears on the foreground, leading to contour–crossing. This review mainly focuses on images consisting of smoothly gray–scaled objects that are separated from each other by smoothly evolving boundaries.

7.4.2 Piecewise Smooth Images

The contours or singularity curves that are present in geometrical images can be seen as lower–dimensional manifolds embedded in a higher–dimensional observation space (the image itself). If we want to focus on the approximation of the contour, we need to concentrate on geometric smoothness (smoothness of the contour) rather than smoothness of the object's surface. For this reason, we define the Horizon class, consisting of piecewise constant functions to measure geometric approximation properties.

7.4.3 Horizon class

The simplest subset of piecewise smooth functions we use for benchmarking is the horizon class $\mathcal{H}$ introduced by Donoho (1999). These objects are constants except for a smooth boundary (with Hölder's regularity $\beta \in (1, 2]$) over the unit square $[0, 1]^2$.

Definition 7.4.1 *(Hölder's regularity). A function $f(x)$ defined on $[0, 1]$ is said to be in the Hölder's class H^β, $\beta > 0$ if there exists a constant $C_\beta < 1$ such that*

$$|D^\alpha f(x) - D^\alpha f(y)| \leq C_\beta |x - y|^{\beta-\alpha}, \quad 0 \leq x, y \leq 1, \tag{7.2}$$

where α is the largest integer not exceeding the Hölder exponent β and D^α denoting the α–th derivative.

The Hölder exponent measures the fractional (Lipschitz) regularity of $D^\alpha f$.

Definition 7.4.2 *Define the Horizon class $\mathcal{H}^\beta$ as the set of functions defined on $[0, 1] \times [0, 1]$ taking the value zero on one side of the Hölder continuous curve $f(x)$ and the value h on the other side.*

$$\mathcal{H}^\beta := \left\{ \gamma : [0, 1]^2 \to \{0, h\} : (x, y) \to \left\{ \begin{array}{ll} h & \textit{if } y < f(x) \\ 0 & \textit{otherwise} \end{array} \right. \right\},$$

with $f(x) \in H^\beta$, $\beta \in (1, 2]$.

Consequently, all image information is located in the boundary $f(x)$.

7.4.4 Piecewise smooth function class

A piecewise smooth image is a function on the unit square characterized by a smooth curve in H^α and two 2–dimensional functions in H^β on either side of the curve.

Definition 7.4.3 *(Piecewise smooth images). $\mathcal{PS}^{\alpha,\beta}$ is the class of functions f such that there exists a curve $y = \gamma(x)$, $\gamma \in H^\alpha$, $\alpha \in (1, 2]$ and two functions $f_1, f_2 \in H^\beta$, $\beta \in (1, 2]$ such that*

$$\mathcal{PS}^{\alpha,\beta} := \left\{ f : [0,1]^2 \to \mathbb{R} \to \left\{ \begin{array}{ll} f_1 & \textit{if } y < f(x) \\ f_2 & \textit{otherwise} \end{array} \right\} \right. .$$

7.5 Numerical Experiments

In this section, it is shown that the proposed scheme in Chapter 3 is sufficient to demonstrate the multiresolution approximation in different resolution level with suitable interpolating basis functions which is Hardy's multiquadric, $\phi(r) = (r^2 + c^2)^{\beta/2}$, where r is the Euclidean distance, c is a suitably chosen constant and β is the tension parameter. The major part of interpolation is that a linear combination of translate of the basis function $\phi : \mathbb{R}^d \to \mathbb{R}$, $\phi(x) = \phi(\|x\|_2)$, such that

$$S_{f,X}(x) = \sum_{k=1}^{N} c_k \phi(\|x - x_k\|_2). \tag{7.3}$$

The basis function ϕ is radial with respect to Euclidean norm $\|\cdot\|_2$.

In this experiment it is assumed that the basis function ϕ, need not to have a compact support and considered the interpolation of the form (7.3). By definition, the approximation is hierarchical method which starts with a decomposition of X into a nested sequence

$$X_1 \subset X_2 \subset \cdots \subset X_{M-1} \subset X_M = X,$$

of M subsets

$$X_k = \{x_1^{(k)}, \cdots, x_{N_k}^{(k)}\} \subset X, \quad 1 \le k \le M.$$

This allows the interpolation to be broken up into M steps. The density of X_k can be measured in various ways, but here it is concentrated on two which between them nicely capture this concept. The first is the separation distance

$$q(X_k) := \min_{1 \le l < m \le N_k} \|x_l^{(k)} - x_m^{(k)}\|/2,$$

which is half the distance between the closest pair of points in X_k. The second is the radius of the largest inner empty sphere

$$Q(X_k) := \max_{x \in \Omega} \min_{1 \le j \le N_k} \|x - x_j^{(k)}\|,$$

where Ω is some fixed compact region in $\mathbb{R}^d$ containing the original X. For convenience, it is assumed that the Ω is the closed interior of some polygon $\partial\Omega$ surrounding X which could, for example, be its convex hull. The measure of q and Q helps to remove the nodes in progressive reconstruction. Removal nodes can be defined as follows.

Definition 7.5.1 *If $x \in \partial\Omega$ is boundary node it has precisely two boundary neighbors x_1, x_2. In this case define*

$$d_{\min}(x) = \min_{i=1,2} \|x_i - x\|, \quad d_{\max}(x) = \max_{i=1,2} \|x_i - x\|.$$

Definition 7.5.2 *A node $x \in \partial\Omega$ is removable if*

- $d_{\min}(x) \le d_{\min}(y)$ *for all* $y \in \partial\Omega$, *and*
- $d_{\max}(x) \le d_{\max}(y)$ *whenever* $d_{\min}(x) = d_{\min}(y)$.

However, the stability of the interpolation process is intimately related to the size of the smallest eigenvalue of the corresponding collocation matrix $A_{X,\phi}$, consequently depends on the separation distance

$$q = \min_{1 \le j < m \le N_k} \|x_j - x_k\|/2,$$

of the set X.

Pointwise error estimates of the interpolation method are usually to be obtained in terms of a local density measure around each of the points $x \in \partial\Omega$. To be more precise, for some positive r this pointwise density measure is given by

$$h_r(x) := \sup_{y \in B_r(x)} \min_{1 \le j \le N} \|x - x_j\|,$$

and is subject to $h_r(x) \leq h$ for all $x \in \partial\Omega$ with some positive constant h which does not depend on x. Then, by previous definition of

$$Q := \max_{x\in\partial\Omega} \min_{1\leq j\leq N} \|x - x_j\|,$$

establish an appropriate relation to h, since

$$Q := \lim_{r\to 0} \max_{x\in\partial\Omega} h_r(x).$$

It is believed that this method would be useful in practical applications for interpolating general scattered data sets. In analogy with hierarchical methods based on regular grids, $(N_k)_k$ to be geometric sequences.

7.5.1 Sampling Operator

A digital image is a rectangular grid of pixels, where each pixel bears a colour value or a gray–scale luminance. We restrict the following discussion to gray–scale images. The digital image can be viewed as an element $I \in \{0, 1, \cdots, 2^{r-1}\}^X$, where X is the set of pixels, and where r is the number of bits in the representation of the luminance values. In this dissertation, images are considered as functions over the convex hull $[X]$ of the set of pixels X, so that $[X]$ constitutes the rectangular image domain. Each pixel in X is corresponding to a planar grid point with integer coordinates lying in $[X]$.

This section focus on discretization of digital images and its interpolation by using normal multiresolution approximation using Hardy's multiquadric functions. A digital $n \times n$ gray–scale image f is a regular arrangement of pixels or picture elements. Pixels are disjoint rectangular areas $\square_{r,s} := [0, n^{-1}] \times [0, n^{-1}] + (r.n^{-1}, s.n^{-1})$ having an integer gray value $f_{r,s}$. Use the shorthand $v_{r,s} := (r, s, f_{r,s})$ to represent a pixel and call (r, s), the location of the pixel. The location of a pixel $v_{r,s}$ is denoted by (r, s). A pixel is the discrete version of a vertex and both terms will be used interchangeably. The major difference is that this scheme uses sampling rather than prediction.

Quasi–Monte Carlo simulation is the traditional Monte Carlo simulation but using quasi–random sequences instead (pseudo) random numbers. These sequences are used to generate representative samples from the probability distributions that we are simulating in our practical problem. The quasi–random sequences, also called low–discrepancy sequences, in several cases permit to improve the performance of Monte Carlo simulations, offering shorter computational times and/or higher accuracy. The essential characteristic of the Monte Carlo method is the use of random sampling techniques to reach a solution of the physical problem.

The generation of (quasi or pseudo) random numbers is a way to generate representative samples (a lot of scenarios) to describe the uncertainties of our physical problem through probability distributions. The uniform $[0, 1]$ distribution permits to generate all the distributions that we need to perform the simulations. Monte Carlo simulations is the strongest application of quasi-random sequences, for example, the independence along the sample path of a stochastic variable. For problems that we need to simulate the entire path we have a multi–dimensional problem.

The uniform distribution in the interval $[0, 1]$ is, for practical purposes, the only distribution that we need to generate for our simulations. The reason is that the samples from the other distributions are derived using the uniform distribution. The uniform distribution can be used to generate either pseudo random number or quasi–random numbers, and algorithms are available to transform a uniform distribution to any other distribution.

The worst–case for Quasi-Monte Carlo (QMC) is much inferior to the crude Monte Carlo with pseudo random numbers. However, the best QMC case is always better than MC. The worst-case bound is not very reliable for practical purposes and, in high–dimensional problems, the reader needs to be aware of the too large required value for N that makes the QMC's worst–case better than the crude Monte Carlo.

Quasi–Monte Carlo methods are valid for integration problems, but may not be directly applicable to simulations, due to the correlations between the points of a quasi–random sequence. The improved accuracy of quasi–Monte Carlo methods is generally lost for problems of high dimension.

7.5.2 Halton sequence

The van der Corput sequence is the basic one dimensional low discrepancy sequence. The van der Corput base 2 sequence is also the first dimension of the Halton sequence, the most basic low discrepancy sequence of our interest, which can be viewed as the building block of other low discrepancy sequences. Halton (1960), Faure (1982), Sobol (1967), and Niederreiter (1987) are the best known low–discrepancy sequences. The construction process of new low–discrepancy sequences involves sub–dividing the unit hypercube into sub–volumes (boxes) of constant volume, which have faces parallel to the hypercube's faces. The idea is to put a number in each of these sub–volumes before going to a finer grid, i.e., nested sequence of points, needed in multiresolution analysis.

In our more basic real option problem, with one source of uncertainty, we can think the number of dimensions d as the number of discrete time (space) intervals of one sample path, so $d = T/D$, where T is typically the expiration of our real options or the horizon of interest, and where D is the base. The number of iterations N is the number of sample paths. So, across the paths at one specific time instant, we want a good uniform sample numbers in order to generate for example a good Normal distribution of a Brownian motion, that probably will result in a Log–normal distribution. The main challenge for the low discrepancy sequences is to avoid the multi–dimensional clustering caused by the correlations between the dimensions. We wish to generate low discrepancy sequences with no correlation for every pair of dimensions.

In statistics Halton sequences are sequences used to generate points in space for numerical methods. Although, these sequences are deterministic they are of low discrepancy, that is, appear to be random for many purposes. The Halton sequence is constructed according to a deterministic method that uses a prime number as its base. The Halton sequence uses one different prime base for each dimension. For the first dimension uses base 2, for the second dimension Halton uses base 3, for the third dimension uses base 5, and so on.

7.6 Experimental Results

We believe that these Halton points sufficiently describe normal multiresolution points sequence in gray–scale images. Then we assume that the Halton point sequences are used in approximating gray–scale images. The following types of images are used to illustrate how each method models geometric and functional smoothness:

1. Horizon images: used to reveal a scheme's ability to capture geometric content,
2. Piecewise smooth images: indicate the ability to approximate both geometric as well as functional content,
3. Natural Scene images : to reveal the capability to approximate less smooth regions.

It is used Lebesgue–based metrics to measure approximation error, although they are not suited to measure geometric resemblance. Ideally, it should be a metric that is closer to the human visual system and incorporates correlation of error locations, but the debate on image quality assessment is still an ongoing research topic and beyond the scope of this dissertation. For the L_2 norm we express approximation quality of $\tilde{f}$ with respect to f in terms of peak signal to noise ratio (PSNR),

$$PSNR = 10\log_{10}\left(\frac{max(f)}{MSE}\right), \quad MSE = \frac{1}{N}\sum_{i=1}^{N}|f_i - \tilde{f}_i|^2.$$

Define an image as a square $S = [0,1] \times [0,1]$. Let us denote the function $H(x)$ in S as the horizon, that is, any one dimensional, continuous and appropriately smooth function defined on the interval $[0,1]$. Such function, for our further purposes, must fulfill appropriate Hölder regularity conditions.

Hence, any horizon function $H(x)$ must fulfill conditions $H \in Holder^1(C_1)$ and $H \in Holder^{\alpha}(C_{\alpha})$ which measure a degree of fractional regularity of H and derivative H' respectively. In other words, it assures appropriately high smoothness of the function. Consider the two dimensional characteristic function

$$f(x_1, x_2) = 1_{x_2 \geq H(x_1)} \quad 0 \leq x_1, x_2 \leq 1.$$

(a) Original Image (b) 30.14dB at 0.1831

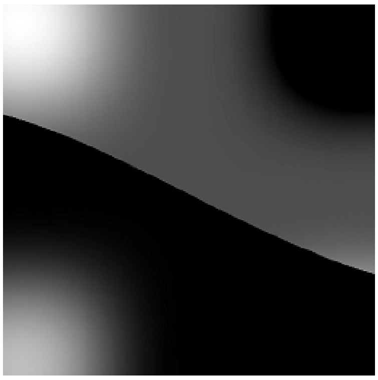
Figure 7.1: Horizon Class Image Constructed with NMT Interpolation (MQF)

(a) PSNR of 37.6dB (b) PSNR of 29dB

Figure 7.2: Wedgelet and Normal Approximation respectively

Let f be a function in $H(x)$ then we call f is a horizon. The function models a black and white image with a horizon, where the image is white above the horizon and black below. Experiments start with the following Horizon Class image of 128×128.

Figure 7.1 shows the results of the experiments on an horizon image. As expected for the wavelet based (frequency) coder, it is noticed ringing artifacts around the contour. There are however localized artifacts in the vicinity of the contour which can be accredited to discretization errors. Several of these errors can be circumvented by careful implementation optimizations at pixel–level.



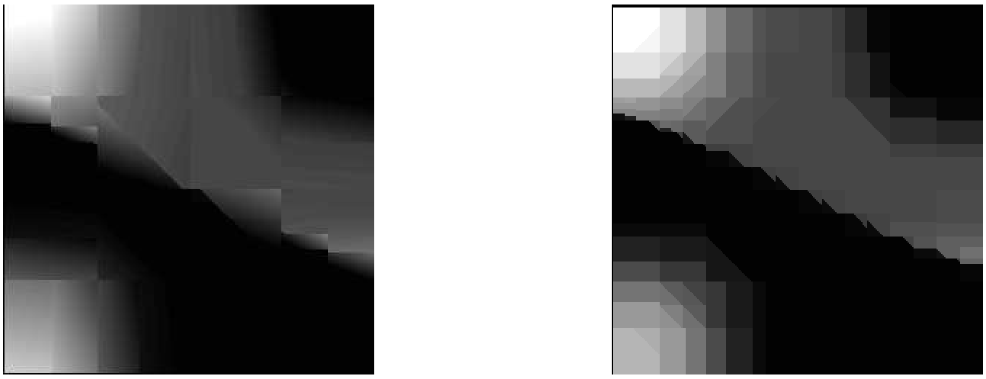
(a) Original Image

(b) 29.54dB at 0.01bpp MQF

Figure 7.3: Piecewise Smooth Image Constructed with NMT Interpolation (MQF)

Wedgelets were used at the very beginning to approximate the special kind of smooth images with one continuous smooth line of discontinuity. The class of such images is called as the simple horizon class. For a fair comparison, it is the horizon image since the wedgelet dictionary contains piecewise constant elements, for approximately the same amount of distortion (PSNR). We observe that wedgelets create a piecewise affine approximation of the contour, while the normal mesh based scheme constructs more fluent approximations by adding continuity to the contour approximations. Since wedgelets construct box splits independently from adjacent boxes (terminal squares), the end points of the wedges do not have to join at the boundary of the terminal squares. Moreover, both endpoints of each wedge have to lie on the border of a dyadic square which restricts the placements of the knots (x and/or y coordinate dyadic) of the piecewise linear approximation of the contour. In contrast, the normal multilevel triangulation (NMT) method is an edge refinement method committing two neighboring triangle leaves at the same scale to have their edges joined at the vertices of the common edge.

Wedgelets do not allow for geometric multiresolution representations. Indeed, the geometry is only modeled at the terminal leaves of a Recursive Dyadic Partitioning tree. Hence, when retrieving a representation on a coarser scale the geometric information will be lost and the transform degenerates to a hereditary constrained Haar transform. In contrast, normal mesh based methods start driving their segment splits towards the contour from



(a) Wedgelet approximation (constant) (b) Wedgelet approximation (linear)

Figure 7.4: Wedgelet approximation: piecewise constant and piecewise linear models. PSNR 30dB.

the most coarse scale. Although, wedgelets have a nearly optimal convergence behavior with respect to horizon images, the performance decreases when more than one contour is present.

The theory of wedgelets was applied with success to many areas of digital image processing. It was used in multiresolution compression of images by Wakin et al. (2003). As shown in (Wakin et al., 2003) the compressor based on wedgelets even gives better performance in image compression than the recognized standard of JPEG2000. Wedgelets has also been used successfully in image segmentation and noise removal. For the piecewise smooth setting, it is used both the original wedgelet approximation using piecewise constant models and piecewise affine models (platelets).

The application of wedgelets considered in this dissertation is only image coding, especially compression. Hence, it has been counted the number of all possible dyadic edges which form the dyadic dictionary of edges present in an image. Of course, as one can easily see, this dictionary does not contain all possible edges, which may occur in an image (it is easy to see that the number of all such edges equals to N^4 (number of edges) which is far larger than M (size of the image)). But, this dictionary allows to represent any smooth image with high accuracy. It follows from the fact that the dictionary contains edges in many locations, scales and orientations.

Assume that considered edges are not degenerated, that is, they do not lie on the border of the square. Then each such edge splits any dyadic square S into two pieces. Let us consider the one of the two pieces which is bounded by lines connecting in clockwise direction from the upper right corner the first of the two edge's vertices and then the second one. Define then the indicator function of that piece. Such function we call wedgelet defined by the edge and denote by $w_{S,m}$. The set of wedgelets of any S is defined as

$$W(S) = \{1\} \bigcup \{allpossible\ w_{S,m}\}.$$

Finally, let us define the Wedgelet Dictionary W as the sum of all sets $W(S)$ of all dyadic squares $S(k_1, k_2, j)$, $0 \leq k_1, k_2 < 2^j$, $0 \leq j \leq J$, where J is a sufficiently large positive integer. Note that such set, similarly as in the case of edge's set, contains wedgelets in many locations, scales and orientations.

The Figure 7.4 shows visual artifacts with constant and linear wedgelet coding schemes at low bit rate. The Figure 7.3 clearly show that NMT Interpolation scheme is superior than the wedgelet approximation models in terms of compression and PSNR. The ringing artifacts could be solved by carefully selecting triangles and the corresponding Halton points. Further to the above the following moon surface image is used to construct an image with NMT Interpolation. The moon image is an ideal sample image to describe piecewise smooth function in horizon class. The Figure A.1 sufficiently demonstrate the use of NMT Interpolation scheme over piecewise smooth images.

Two gray–scale images Cameraman and Lena with 256×256 pixels are selected for this experiment for natural scene images. These images are discrete sets of pixels. In order to find normal piercing points, we need to interpolate these pixel matrices. The Cameraman image is more geometric than Lena. A trivial triangular mesh allows for a piecewise planar interpolation in each point. As a consequence, there is no real discontinuity, only steep transitions. Many edges in images are blurred over several pixels anyway. The special actions to deal with real discontinuities are therefore unnecessary in this practical example.

The NMT Interpolation scheme is applied on both images. The following figures shows that the NMT Interpolation performs well in the Lena image than the Cameraman image. Ringing effects is observed at the geometric boarder of the Cameraman image. This could be solved by carefully selecting interpolating points. Moreover, inset of each images with 128×128 pixels are encoded with the same NMT Interpolation scheme. For small structures, such as the eyes in the photograph, or for texture, wavelets perform better in filling up the details. These results are compared with other schemes namely JPEG2000 and GPP. Inset of the cameramen image Figure A.2 and Lena image Figure A.4 clearly shows that NMT Interpolation scheme also performs well in small structures, such as the eyes in the photograph, or for texture regions in images, at low–bit rate than the JPEG2000. Figure A.5 shows that the NMT Interpolation scheme performs well than the popular image coder JPEG2000 at low–bit rate and the visual effect is far better than geometric piecewise polynomial approximation. It is important to observe that the trade off between bit–rate and peak to signal to nose ratio (PSNR) achieved using NMT Interpolation is better than the other popular image coding schemes.

In practice, images are obtained as samples on a square grid, hence using normal meshes is equivalent to a remeshing operation. A second inverse remeshing would be necessary to display a normal mesh approximation using a conventional displays. This makes things more complicated in practice. But, NMT Interpolation scheme does not require such remeshing. This is a clear advantage over the NMT approximation.

7.7 Summary

In this chapter, the performance of the normal multiresolution approximation is compared with recent geometric approximation schemes. Although, there are no practical wedgelet image coders available yet, the experiments show that the normal approximation schemes proposed in this dissertation is a valid competitor in image coding schemes. When compared to wedgelets, the normal approximation schemes produce more fluent approximations of the geometric content, resulting in more visually pleasing approximations.

NMT Interpolation scheme compress natural scene images more effectively than the

JPEG2000 standard. Further compression is also possible by eliminating redundant nodes of interpolation.

This dissertation is concluded with the following chapter where it is discussed the outcome of this research and make some comments on further research.

Chapter 8

Conclusion and Further Research

This dissertation is concluded with a discussion on the results and an outlook on possible further research. The central result in this thesis is the study of necessary and sufficient conditions on normal wavelet bases for $C(\Omega)$, where Ω is a bounded and simply connected domain of $\mathbb{R}^d$, $d \geq 1$, to constitute bases in less smooth spaces, such as Besov spaces $B^{\alpha}_{p,q}(\overline{\Omega})$, $1 \leq p, q < \infty$, $1 < \alpha < 1$, (Theorem 6.3.1). The underlying concept is that how normal multiresolution approximation works with gray–scale images. Thus, wavelet coefficient decay can be characterized in terms of membership in a suitable sequence space. More precisely, the arising coefficients are in $b^{\alpha}_{p,q}$ if and only if the function s is in the Besov spaces, $B^{\alpha}_{p,q}$. Moreover, triangulation in multivariate setting, that is, the surface is not too flat, is effectively solved with the multiquadric function. This is an important consequence of this scheme.

In Chapter 3, it is established that the interpolating basis function for normal multiresolution approximation using Hardy's multiquadric function $\varphi(x) = (x + \lambda)^{\beta/2}$. The remarkable fact is that the basis function span shift–invariant spaces. Based on the linear combination of the basis function form the shift–invariant space, the approximation properties of normal multiresolution approximation are established in Sobolev spaces which is:

$$\|f - S_f\| \leq C\|f\|_{W^k_2(\Omega)} h^{k+1},$$

if $h \leq h_0$ with h defined as

$$h := \sup_{x \in \Omega} \min_{1 \leq j \leq N} \|x - x_j\|.$$

In Chapter 4, the interpolating basis function of multiquadric has been generalized with localization process in order to have compact support and then realized as quasi–interpolating wavelet on real line, where quasi–interpolating wavelet defined on the principle of one–point quasi–interpolating function which generate a scale space. The construction of an quasi–interpolating basis is built to resemble the construction of non–orthogonal wavelet bases on real line, by making simple assumption that wavelets are simply higher resolution quasi–interpolating scaling function. However, this is just one way to construct quasi–interpolants. There is a dyadic construction of an interpolating splines of Schauder basis of $C[0, 1]$ by Domsta (1976) for which function space characterizations have been proven; but Domsta's basis does not have dilation and translation homogeneity.

The quasi–interpolating basis constructed in this dissertation is shift invariant and scale invariant in $L_1(\mathbb{R})$. This has been realized as triangulation in a bounded domain. Thus, the wavelet decomposition considered as linear splines on triangulation at different resolution level with subdivision connectivity. Moreover, pointwise convergence property of the wavelet transform is explored.

In Chapter 5, the triangulation property of the quasi–interpolating basis function, analog to Schauder basis, is characterized in a multivariate domain Ω with the assumption that the boundary $\partial\Omega$ is closed set that preserving Markov's inequality, which resolve the problem of basis function is not piecewise linear in multivariate settings, i.e., the quasi–interpolating wavelet in the previous chapter is realized in multivariate setting with some additional assumptions. An important assumption is that the open bounded domain satisfying uniform cone condition. Then with the assumption on boundary of the domain satisfying minimal smooth condition the quasi–interpolating wavelet is extended to Euclidean space. The multiresolution property of the basis function is defined in terms of multiresolution property of the domain on which the basis function is constructed.

Further, an admissible triangulation property is introduced in this chapter in order to have a fractal nature of wavelet decomposition, which implies a d–set on $\partial\Omega$. An important advantage is that the the basis function is boundary free as opposed to other construction of wavelet basis functions.

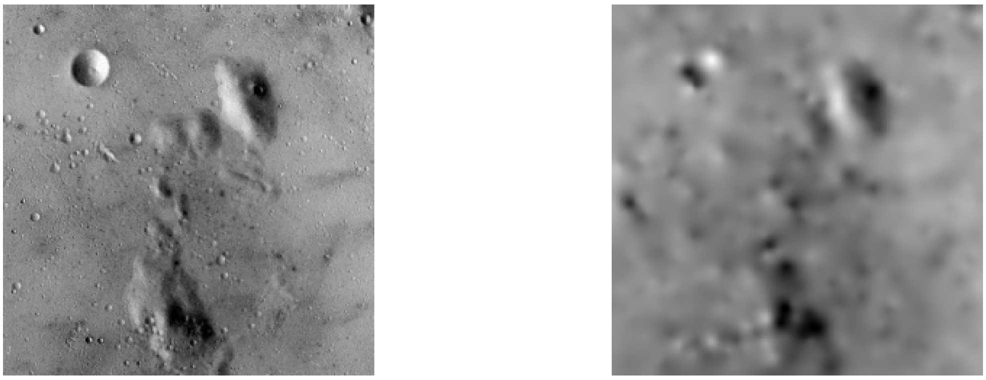
In Chapter 6, function with less smoothness are decomposed in Besov spaces based on the normal wavelet basis function. Thus, the normal wavelets are realized in less smooth spaces. In this chapter much of the work is similar to Jonsson and Kamont (2001) except some additional assumptions that the normal wavelet decomposition produces admissible triangulation and the boundary of the domain admits some regularity conditions.

Based on the above decomposition, a compressing scheme is presented in Chapter 7 which is optimal in the sense of nonlinear approximation i.e., normal wavelets may be quantized with nonlinear approximation as an optimal procedure for image compression. This may be further extended in $BV(\Omega)$ with the characterization of $B^1_{1,1}(\Omega)$, since bounded variation space is best suited for gray–scale modeling.

Recently, a framework for building natural multiresolution structures on manifolds was introduced by Coifman and Maggioni (2004), that greatly generalizes, among other things, the construction of wavelets in Euclidean spaces. This allows the study of the manifold and of functions on it at different scales, which are naturally induced by the geometry of the manifold. Hence, concept of normal wavelets could be studied in the context of natural multiresolution structures on manifolds.

Appendix A

Test Images



(a) Original Image

(b) Constructed Image 24.0047dB at 0.0107bpp

Figure A.1: Moon Image constructed using NMT Interpolation Scheme.

(a) Original Image

(b) 17.94dB at 0.01bpp

(c) Inset 25.0411dB at 0.17bpp

(d) Inset 27.3778dB at 0.17bpp

Figure A.2: Cameraman Image constructed with NMT Interpolation (MQF).

(a) 20.1dB at 0.45bpp JPEG2000

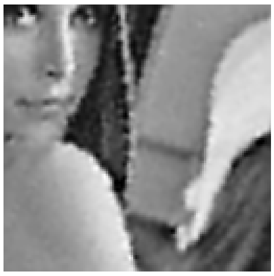
(b) 21.5dB at 0.05bpp GPP

Figure A.3: Cameraman Image constructed with JPEG2000 and Geometric Piecewise Polynomial Approximation.

(a) Original Image

(b) 18.3dB at 0.01bpp MQF

(c) Inset 26.9404dB at 0.17bpp



(d) Inset 29.2475 at 0.17bpp

Figure A.4: Leena Image constructed with NMT Interpolation (MQF).

(a) 25dB at 0.02bpp JPEG2000

(b) 25dB at 0.02bpp GPP

Figure A.5: Lena Image constructed with JPEG2000 and Geometric Piecewise Polynomial Approximation.

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